INSTITUT FÜR INFORMATIK
DER LUDWIG–MAXIMILIANS–UNIVERSITÄT MÜNCHEN

Bachelor Thesis

Theoretical study of photon mapping with stratification

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Hiermit versichere ich, dass ich die vorliegende Bachelorarbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

München, den 22. Juli 2020

Kathrin Hartmann

(Unterschrift des Kandidaten)

Abstract

In the last decades computer-generated images have become indispensable. From product design over the game and film industry until medical analysis - computer graphics plays a significant role in all of these sectors. The photon mapping algorithm is a relevant method used for generating especially photo realistic images. In this work the bias of photon mapping in combination with stratified sampling is examined. For this purpose three types of scenes are considered: a one dimensional setup, surface illumination and volumetric effects. The theoretical study of all three setups detects an overestimation bias similar to the bias described in "Describing and Solution of an Unreported Intrinsic Bias in Photon Mapping Density Estimation with Constant Kernel". This overestimation bias is verified afterwards with an experimental study of stratified photon mapping. With different estimates for calculating the irradiance that were used in other theoretical studies of photon mapping it is investigated how the detected overestimation bias for stratified photon mapping behaves. The estimate proposed in [Gar12] leads to a small underestimation bias while the estimate from [Jen02] reduces the absolute value of the bias, showing a mixture between over- and underestimation. In the end it is presented scientific research that applies the theoretical concept behind photon mapping and it is proposed a study of variance in combination with stratified photon mapping for a future research area.

Contents

1. Introduction

The purpose of this work is to extend the theoretical framework designed by García et al. in the paper "Description and Solution of an Unreported Intrinsic Bias in Photon Mapping Density Estimation with Constant Kernel" by combining photon mapping with stratified sampling and to show how stratification affects the overestimation bias of the photon mapping algorithm.

Section 1 gives an introduction on computer graphics, photon mapping and related work, including also mathematical and statistical basics that are important to understand the next chapters. Section 2 shows the empirical study of photon mapping with stratified samples and and the resulting bias. Afterwards, in section 3 this bias is verified experimentally. In order to reduce the bias two different historical approaches are applied to stratified photon mapping in section 4. Eventually, section 5 presents applications of the concept of photon mapping and a proposal for future work.

1.1. Computer graphics

"A picture is worth a thousand words" - a single image can tell much more than a text. This could be the reason why computer graphics has become an indispensable part of our daily lives. Computer-generated images accompany us all day long: They entertain us while we play games on computers. They inspire us when we come across an animated video on our smartphones or tablets. They thrill us every time we watch films with special effects on television [Sal11]. Beyond that, computer graphics also plays an important role in areas where we would not suspect it in first place. The design sector needs computer graphics for generating models of the product before producing it. Simulations for architecture and urbanism are generated by computers. Beyond that, in medical areas the visualization of data in form of computer-generated images helps to detect diseases [Sal11].

The more realistic a computer-generated should be, the more resources for computation are needed and the more preparation time is required. This is the so called speed - quality trade-off which one inevitably comes across when using computer graphics. Basically, computer graphics deals with creating a two-dimensional image from a three-dimensional geometric model. To make the outcome seem most realistic, there are several steps necessary: After first arranging a scene with the geometric models, a camera perspective, shadows, light sources and colour have to be added. This process is called rendering. In the end, the scene has to be animated by adding motion effects to the objects [Car15]. In photorealistic rendering the image should be rendered as realistic as possible. This means that the rendering process has to be optimized in order to make the final outcome look like a photography.

For simulating light in computer generated images there can be distinguished two kinds of illumination techniques. Local illumination simulates only light coming directly from the light source for illuminating an object. Global illumination instead simulates the light transport along the scene. Thus for calculating the light intensity of a point x, additionally to the light from a light source illuminating x directly, also indirect illuminations from reflected light rays are included in the light value of x.

1.2. The photon mapping algorithm

The photon mapping algorithm has a significant meaning for photorealistic rendering. It is capable of simulating global illumination effects such as reflections and caustics [Dut06]. Photon mapping steps in already before rendering the scene. In a way, photon mapping is similar to the corpuscular theory of light in physics: A light source emits small energy packets, so called photons, with uniformly distributed power in random directions into the scene. When the photons collide with objects on the scene, they behave according to the material that the objects consist of – they are absorbed, reflected ore refracted. Reflected and refracted photons are moving onwards in the scene until hitting a diffuse surface while hits from absorbed photons are stored in a photon map. When afterwards rendering the scene, the radiance of a surface point x can be estimated by looking up all k nearest photons in the photon map that are within a certain radius r_k and considering their energy [Gar12] [Shi03]. This process is called nearest neighbour density estimation.

1.3. Mathematical and statistical prerequisites

For the theoretical study of the photon mapping algorithm later on it is necessary to be familiar with a few mathematical concepts. We will need density estimation and order statistics for the photon mapping algorithm, the expected value of a random variable to test the calculation of the irradiance and of course stratified sampling.

1.3.1. Expected value of a random variable

Generally spoken, the expected value of a random variable X is the probability-weighted average of all its possible values. In other words the expected value is calculated by multiplying each of the possible outcomes x_1, x_2, ... x_n of X by the likelihood each outcome will occur and then summing all of those values [Wei20].

$$E[X] = \sum_{i=1}^{n} P(x_i)x_i \tag{1.1}$$

One could expect that $E[\frac{1}{X}]$ equals $\frac{1}{E[X]}$. This is not true and can be disproved with a simple example. We are looking at a coin flip with an ideal coin. In the sample space are two outcome values, 1 and 2. As the coin is ideal, we expect both values to appear with the same probability $\frac{1}{2}$, so their expected value is $E[X] = \frac{1+2}{2} = \frac{3}{2}$. The inverse of the expected value is $\frac{1}{E[X]} = \frac{2}{3}$.

Taking instead the inverse of the outcome values and then calculating the expected value of the inverses, it results in a different value: $E[\frac{1}{X}] = \frac{1^{-1}+2^{-1}}{2} = \frac{3}{4}$. As $\frac{3}{4} \geq \frac{2}{3}$, also $E[\frac{1}{X}] \geq \frac{1}{E[X]}$. Overall it is valid that for every convex function ρ, $E[\rho(X)] \geq \rho(E[X])$ [Jen06].

1.3.2. Density estimation

Density estimation is the construction of an estimate of the probability density function for given data points [Sil02]. The probability density function f gives a natural description for the distribution of a random quantity X. With the following relation, other probabilities associated with X can be found, for example the cumulative distribution [Sil02].

$$P(a < X < b) = \int_a^b f(d)dx \quad for\ all\ a < b \tag{1.2}$$

1.3.3. Order statistics

Order statistics deals with the probability distribution of elements in ordered lists. The list contains the ordered results of $\tilde{n}$ random experiments: $[x_{(1)} \dots x_{(\tilde{k})} \dots x_{(\tilde{n})}]$. As we use n and k to indicate specific properties later on in the photon mapping algorithm, the variables in the ordered list are denoted with $\tilde{n}$ and $\tilde{k}$. $\tilde{n}$ indicates the number of elements in the list and $\tilde{k}$ the $\tilde{k}^{th}$ elements in the list whose probability distribution is to be calculated. $F(x)$ indicates the cumulative distribution function and $f(x)$ the probability density function. The distribution of the $\tilde{k}^{th}$ order statistic is [Dav70] [Gar12]:

$$f_{X_{(\tilde{k})}}(x) = \frac{\tilde{n}!}{(\tilde{k}-1)!(\tilde{n}-\tilde{k})!} F^{\tilde{k}-1}(x)(1-F(x))^{\tilde{n}-\tilde{k}} f(x) \tag{1.3}$$

1.3.4. Stratification

Stratification is used in statistics as a sampling technique. The objective of stratified sampling is to ensure that all regions of the sample space are covered with samples. It is also used to reduce the overall variance of an estimate by putting more samples in regions where the values of the probability density functions are likely to vary a lot [Gen05]. This is how it works: Stratified sampling is a combination of random and systematic sampling. To create a stratified sample, the sampling frame has to be partitioned into several blocks, based on a relevant variable. After that random samples are drawn within these blocks [Dad17].

1.4. One-dimensional photon mapping with random samples

This is a model of one-dimensional photon mapping density estimation according to [Gar12]. We take the interval $[-1, 1]$ as our scene that should be illuminated. The light source distributes n photons randomly on this interval (Figure 1.1 (a)). The photon map query is performed at the midpoint $P = 0$ and should consider its k nearest photons for calculating the irradiance. For doing so, the photons first have to be ordered by their distance to P: from the 1^{st}, 2^{nd}, 3^{rd} ... k^{th} until the n^{th} nearest photon (Figure 1.1 (b)). The distance from P to the k^{th} nearest photon is r_k (Figure 1.1 (c)). As we perform the query at the midpoint of the interval, the value of r_k lies in between $[0, 1]$.

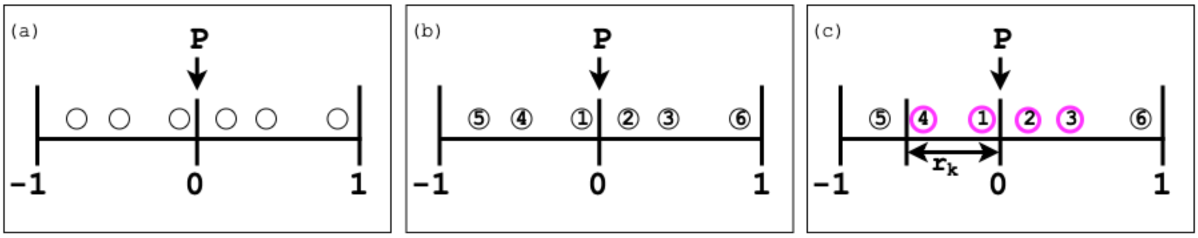


Figure 1.1.: Example for n = 6 and k = 4.
(a): The interval [0, 1], query point P and the six randomly distributed photons;
(b): The photons ordered by their distance to P;
(c): The distance r_k from P to the k^{th} nearest photon relative to P

In order to simulate illumination in the model, there has to be a value that describes the radiance of the light in the scene: The irradiance $I()$ is defined as the value of the incoming flux of radiation per interval unit. It is constant, so all points in the interval have the same irradiance. $I(P)$ is the irradiance at the query point P [Gar12]. We denote the irradiance with the symbol $I()$, like it was done in [Gar12], in order to avoid confusion with the expected value, denoted $E[]$.

Every photon carries a certain amount of energy, the flux ϕ. To calculate it, we multiply the irradiance at one point P of the interval with the interval length, which is 2 for the interval $[-1, 1]$. Dividing the flux among the photons it results in [Gar12]:

$$\phi = \frac{2I(P)}{n} \tag{1.4}$$

The estimate of the irradiance $\hat{I}$ on a distance r_k to the photon map query point takes the estimate of the incoming flux of all k nearest photons and divides it by the interval length that covers all k nearest photons, which is twice the distance of the k^{th} nearest photon, r_k [Gar12].

$$\hat{I}(r_k) = \frac{k\phi}{2r_k} \overset{\text{with (1.4)}}{=} \frac{kI(P)}{r_k n} \tag{1.5}$$

Now, we want to see how well this estimate works. Therefore, we calculate the expectation of the estimate of the irradiance according to [Gar12]. In general, the expected value of a random variable X whose cumulative distribution function admits a probability density function $f(x)$ is:

$$E[X] = \int_{\mathbb{R}} x f(x) dx \tag{1.6}$$

In our case, X is the estimate of the irradiance $\hat{I}(r_k)$. As the absolute distance from P to the k_{th} nearest photon in the interval $[-1, 1]$ lies between 0 and 1, it is integrated from 0 to 1. As we order the photons by their distance to the query point, we need to take care of something else: The probability distribution of ordered elements does not only depend on their probability density function but also on their position in the sampling frame. That is why we take $f_{X_{(k)}}(r_k)$, the distribution of the $\tilde{k}^{th}$ order statistics, as our $f(x)$ in equation (1.6). The resulting equation is:

$$E[\hat{I}(r_k)] = \int_0^1 \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k \tag{1.7}$$

We have a look at the distribution of the $\tilde{k}^th$ order statistics from equation (1.3). For photon mapping with random samples, n photons are thrown under the same conditions on the interval. This means, $\tilde{n}$ equals n. $\tilde{k}$ is equal to k as we are interested in the k^{th} nearest photon.

Furthermore, we need the probability density function f and the cumulative distribution function F. The probability of finding the k^{th} nearest photon within the distance $[0, 1]$ from P is 1. With density estimation from equation (1.2) we know that the probability density function f can be found with $\int_0^1 f(r_k) dr_k = 1$ so it must be:

$$f(r_k) = 1 \tag{1.8}$$

The cumulative distribution is the integral of f over the whole scene interval:

$$F(r_k) = \int_{-1}^{1} 1 = r_k \tag{1.9}$$

Putting everything together, the distribution of the k_{th} order statistics for photon mapping with random samples results in:

$$f_{X_{(k)}}(r_k) = \frac{n!}{(k-1)!(n-k)!} r_k^{k-1} (1 - r_k)^{n-k} \tag{1.10}$$

Now, we are ready to calculate the expectation of the estimate of the irradiance with equation (1.7):

$$\begin{aligned} E[\hat{I}(r_k)] = \int_0^1 \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k &= \int_0^1 \frac{kI(P)}{r_k n} \frac{n!}{(k-1)!(n-k)!} r_k^{k-1} (1 - r_k)^{n-k} dr_k \\ &= \frac{I(P) k \Gamma[k-1]}{\Gamma[k]} = I(P) \frac{k}{k-1} \end{aligned} \tag{1.11}$$

With the result $I(P)\frac{k}{k-1}$ it is possible to calculate the relative error of the photon mapping irradiance. We visualize it in figure 1.2 for $k = 200$. This empirical approach can be compared with an experimental implementation of the photon mapping algorithm in a C program. We set n to 100 000, chose k within the interval from 2 to 200 and repeated the experiment ten thousand times. The relative errors from both approaches are fitting well together (Figure 1.2). The graph shows the overestimation bias that was found in [Gar12].

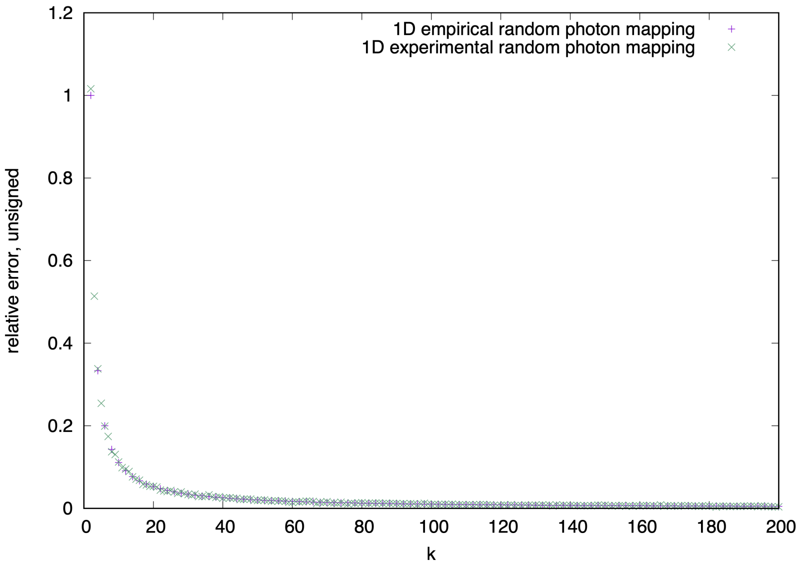


Figure 1.2.: Plot of the relative error of the irradiance from simulating the photon mapping algorithm in a C program compared to the relative error of the modelling of the 1D case with random samples according to [Gar12].

1.5. Photon mapping with stratified samples

Combining stratification with photon mapping can be realized as follows: Like before there is a scene with an interval from $[-1, 1]$ and a light source, illuminating the scene. The lights source is divided in n subdivisions, emitting the same number of photons out of each subdivision. This stratified emission creates a secondary stratification on the scene which means that the scene interval is divided in n subdivisions with each subdivision containing one photon. So one subdivision has the length $\frac{2}{n}$. The query point P where radiance is estimated is at the midpoint of the interval.

For this approximation we do not yet take into account the k nearest photons around the query point. The radius within which photons are considered for the irradiance has a fixed value r. From the query point P on, photons are searched within the length of the radius. r can be divided into two parts on each side of query point P. The integer part i contains the length of whole subdivisions that are covered by the radius. It is a multiple of $\frac{2}{n}$. In the fraction part f are rest parts that appear if r is not a multiple of the subdivision length.

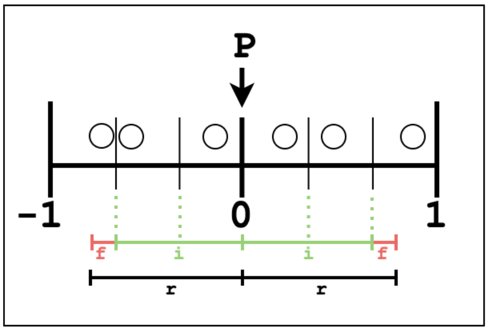


Figure 1.3.: Example for $n = 6$: The radius r is divided in a fraction part f and an integer part i.

It is guaranteed that in each subdivision is situated one photon, so on the integer part we expect as many photons as it covers numbers of subdivisions. That would be $i \cdot \frac{n}{2}$ photons. In the fraction part instead, a photon is only present with a certain probability. f is a fraction of one subdivision, so the probability of finding a photon within f is $f \cdot \frac{n}{2}$ and $1 - (f \cdot \frac{n}{2})$ of not finding one.

For the entire interval $[-1, 1]$ we expect $k = 2 \cdot (i \cdot \frac{n}{2} + f \cdot \frac{n}{2})$ photons. The fixed total area that is covered by the search of photons has a length of $2 \cdot (i + f)$. So the expected number of photons per unit area is:

$$E\left[\frac{k}{2r}\right] = \frac{2 \cdot (i \cdot \frac{n}{2} + f \cdot \frac{n}{2})}{2 \cdot (i + f)} = \frac{n}{2} \tag{1.12}$$

The estimate of irradiance $\hat{I}(P)$ is the expected number of photons per unit area times ϕ (see equation (2.16)):

$$\hat{I}(P) = \frac{n}{2}\phi = \frac{n}{2}\frac{2I(P)}{n} = I(P) \tag{1.13}$$

The last equation shows that the estimate of irradiance at P is equal to the real irradiance at P. This implies that stratified sampling with a fixed value of the radius within photons are considered for the irradiance is working. In the next chapter we will look at stratified samples with normal photon mapping where the k^{th} nearest photon of P determines the radius.

1.6. Previous work on photon mapping

In 1993 the photon mapping algorithm was developed in order to extend the concept of ray tracing for efficiently simulating all types of direct and indirect illumination. It has solved the problem of noise when rendering small specular surfaces and reflections by mirrors and glass. Photon mapping is still used in some of the most famous renderers [Jen03] [Shi03].

The pioneer of photon mapping is Henrik Wann Jensen. In his paper “A Practical Guide to Global Illumination using Photon Mapping” he describes the functionality of the algorithm in its several steps (emission, tracing, storage and rendering of photons) and is simulating scenarios with different light sources. Apart from that he mentions noise that appears when emitting only a small number of photons. His solution for reducing these artefacts is using filters that increase the weight of the photons near the considered surface point. Examples of filters in the paper are the Cone filter, the Gaussian filter and the specialized differential filter [Jen03].

Henrik Wann Jensen also made the first approach for testing photon mapping with an estimate of irradiance in the year 1996. To compute the radiance at a surface point P, he located the k photons with the shortest distance to P and divided their flux by the area of a sphere centered at P that contains these k nearest photons and has radius r [Jen96]. However, this estimate has an overestimation bias, see [Gar12].

In [Gar12] it was discovered an overestimation bias in the photon mapping algorithm itself. Referring to the afore-mentioned functionality of photon mapping, usually there are considered the k nearest photons around a surface point x to calculate its luminosity. It was concluded that the bias could be removed by still including the area that contains that contains the k nearest photons but at the same time adding only the flux of $k-1$ photons to the irradiance [Gar12].

After this detection, it was done further examinations on photon mapping with different kernels. [Gar14] shows two biases for the Gaussian filter. Apart from that, the consistency of the cone filter is refuted and it is proven that the epanechnikov kernel and the silverman kernel are consistent.

Another approach for removing the photon mapping bias is to calculate only half of the flux of the k^{th} nearest photon. This approach was presented in “A Practical Guide to Global Illumination using Photon Mapping” by Jensen in the year 2002. Only half of the power of last photon is included in the photon mapping query. In that way, the irradiance is less biased [Jen02].

2. Theoretical study of stratified photon mapping

In the following sections scenes of different dimensions are tested with stratified photon mapping. First the one-dimensional space, then surface illumination and in the end volumetric effects. In every case it is investigated first a simple model before the case is modelled according to [Gar12].

2.1. Stratified photon mapping in a one-dimensional space

We take the interval $[-1, 1]$ as our scene. The light source has n subdivisions, each subdivision emits one photon. This creates a secondary stratification on the interval. So interval is divided in n subdivisions with one photon each. The position of the photon within the subdivision is random (Figure 2.1). The photon map query is performed at the midpoint $P = 0$ and should consider its k nearest photons for calculating the irradiance.

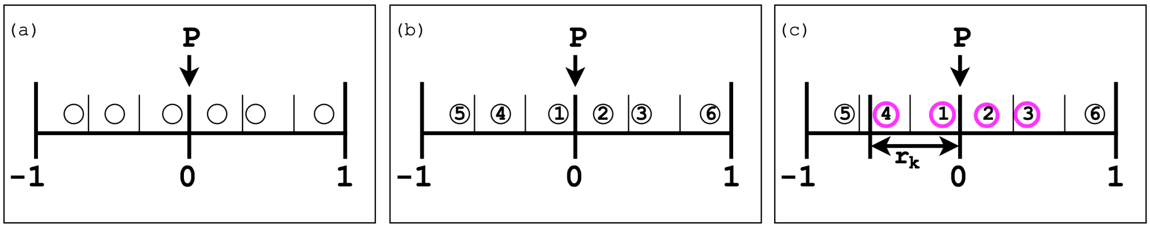


Figure 2.1.: Example for n = 6 and k = 4.
(a): The interval [0, 1], query point P and the six stratified distributed photons;
(b): The photons ordered by their distance to P;
(c): The distance r_k from P to the k^{th} nearest photon relative to P.

As there are no changes in the setup of the scene compared to the random case, only in the way of distributing photons, we use the same estimate of irradiance as before with $I(P)$ as the irradiance at midpoint P:

$$\hat{I}(r_k) = \frac{kI(P)}{r_k n} \tag{2.1}$$

2.1.1. A first approximation in one dimension

In the following approximation we use the characteristics of the expected value of two variables X and Y which randomly take on values from a range [A,B]. When the variable Y is bigger than X, then X will vary from 0 to Y and vice-versa for when X is the bigger. The expected value for the bigger value lies therefore at two thirds of the range and the one for the smaller value at one third [Sta15]. For even and odd n this results in two different cases, that is why we have a look at them separately.

One has to notice that this is only an approximation for photon mapping. As already mentioned in the introduction, $E[\rho(X)] \geq \rho(E[X])$ [Jen06]. In the first approximation it is taken an expected value for r_k and with this value it is calculated the estimate of the irradiance, so $\hat{I}(E[r_k])$. To get the significant expected value it is necessary to compute $E[\hat{I}(r_k)]$, though. This will be done in the following chapter, according to [Gar12].

First approximation for even n

Let us begin with the case where n has an even value. We have a look at the example from figure 2.1 for $n = 6$: There are six subdivisions. Every photon is in his own subdivision. The expected value of the average position of each photon is in the middle of each subdivision (Figure 2.2).

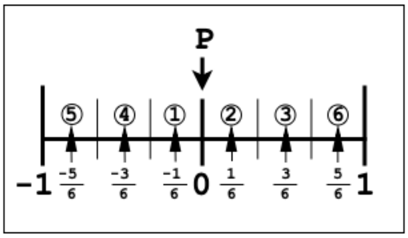


Figure 2.2.: Example for n = 6. The arrows indicate the expected value of the photons.

As the setting is symmetric, it can be simplified. We project the photons in the negative area on the photons in the positive area of the interval. In that way, the photons are ordered by the absolute value of their distance to the midpoint P. Now, there are three subdivisions with two photons each. The expected value of the position of the photons changes: The photon that is situated closer to the midpoint is expected at $\frac{1}{3}$ of the subdivision, the one farther away at $\frac{2}{3}$ (Figure 2.3).

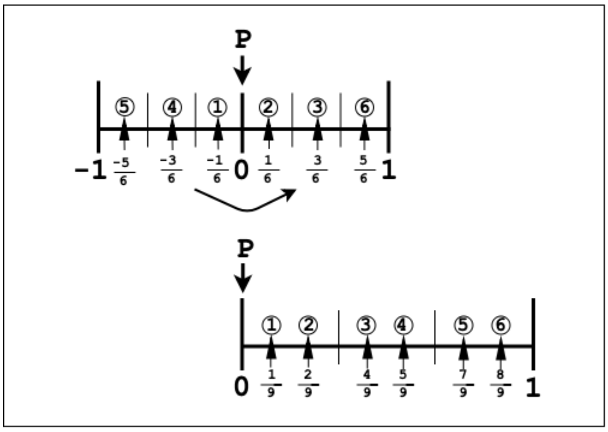


Figure 2.3.: Simplification the example for $n = 6$. The arrows indicate the expected value of the photons.

Photons with even k are situated in the $\frac{k}{2}^{th}$ subdivision counted from P. The subdivision length is $\frac{2}{n}$. When considering each subdivision as a small interval $[A, B]$, the border A of the k^{th} nearest photon would be at $(\frac{k}{2} - 1)\frac{2}{n}$ and the border B at $\frac{k}{2} \cdot \frac{2}{n}$. As the photon with an even value of k is the photon in the subdivision with the larger distance to P, we expect it at $\frac{2}{3}$ of the interval. So its expected value of irradiance is (see Appendix B for details):

$$\hat{I}(r_k) = \frac{kI(P)}{\frac{2}{3}[A, B]n} = \frac{kI(P)}{\frac{3k-2}{3n}n} \tag{2.2}$$

Photons with odd values of k instead are situated in the $\frac{k+1}{2}^{th}$ subdivision. The interval $[A, B]$ has the values $A = (\frac{k+1}{2} - 1)\frac{2}{n}$ and $B = \frac{k+1}{2} \cdot \frac{2}{n}$. They are the ones in the subdivision with the smaller distance to P, so we expect them at $\frac{1}{3}$ of the interval. Their expected value of irradiance is (see Appendix B for details):

$$\hat{I}(r_k) = \frac{kI(P)}{\frac{1}{3}[A, B]n} = \frac{kI(P)}{\frac{3k-1}{3n}n} \tag{2.3}$$

First approximation for odd n

Now, let us look at the case in which n has an odd value. We have a look at an example for $n = 5$:

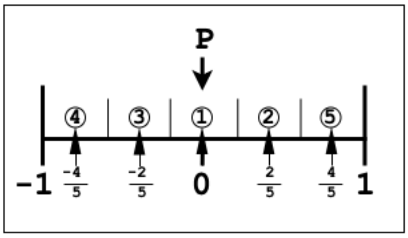


Figure 2.4.: Example for $n = 5$. The arrows indicate the expected value of the photons.

We have now five subdivisions, one photon is situated within each subdivision. The expected value of the average position of the photon is again in the middle of the subdivision (Figure 2.4). Also this setting can be simplified. When projecting the negative part of the interval on the positive part, we get one half subdivision with one photon and two entire subdivisions with two photons (Figure 2.5). In the subdivisions with two photons, the expected value of the photons changes as in the other setting - the nearer to the midpoint situated photon is at $\frac{1}{3}$ of the subdivision, the one farther away situated at $\frac{2}{3}$ of the subdivision.

Photons with odd k are situated in the $\frac{k}{2}^{th}$ subdivision counted from P. The subdivision length is $\frac{2}{n}$. When considering each subdivision as a small interval $[A, B]$, the border A of the k^{th} nearest photon would be at $(\frac{k}{2} - 1)\frac{2}{n}$ and the border B at $\frac{k}{2} \cdot \frac{2}{n}$. As the photon with an odd value of k is for $k > 1$ the one of the two photons in the subdivision with the larger distance to P, we expect it at $\frac{2}{3}$ of the interval. Photons with odd values of k in an environment with odd n are expected at the same place as photons with even values of k in an environment with even n. This means that equation (2.2) is valid also for the expected value of odd n and odd k.

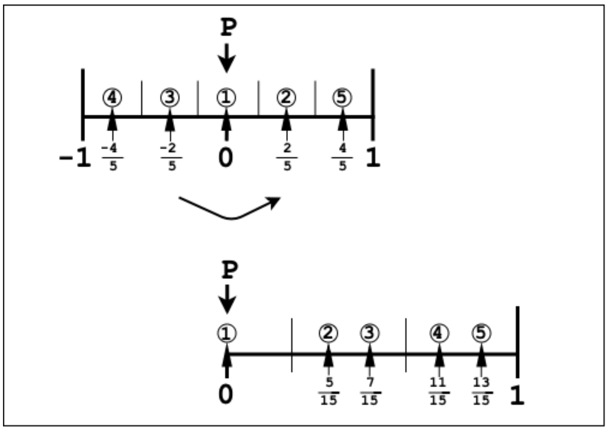


Figure 2.5.: Simplification of the example for n = 5. The arrows indicate the expected value of the photons.

Photons with even values of k instead are situated in the $\frac{k+1}{2}^{th}$ subdivision. The interval $[A, B]$ has the values $A = (\frac{k+1}{2} - 1)\frac{2}{n}$ and $B = \frac{k+1}{2} \cdot \frac{2}{n}$. They are the ones in the subdivision with the smaller distance to P, so we expect them at $\frac{1}{3}$ of the interval. Their expected value of irradiance is the same as for photons with an odd value of k in an environment with even n, so equation (2.3) is valid here.

Results

We have gained two formulas in the first approximation, equation (2.2)) and equation (2.3)). When plotting the results within the interval from 2 to 200, with $n = 100\ 000$ and with $I(P) = 1$, we get the graph in figure 2.6. It confirms that the cases for even n, even k and odd n, odd k are the same and also the ones for even n, odd k and odd n, even k. This means the parity of n and k is important for this model of one-dimensional stratified photon mapping. The graph also shows that there is an overestimation bias for all n and k.

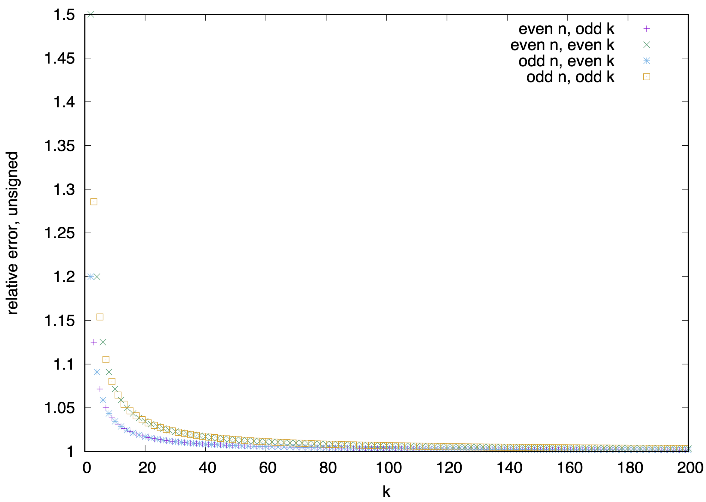


Figure 2.6.: The data points for even n and even k are on the same lines as the ones for odd n and odd k. The data points for even n and odd k are on the same line as the ones for odd n and even k.

2.1.2. One-dimensional stratified sampling with order statistics

Let us test the estimator of irradiance for stratified samples in a one-dimensional setting according to [Gar12], in the same way as it was done in section 1.4 for random samples.

The estimator of irradiance is:

$$\hat{I}(r_k) = \frac{kI(P)}{r_k n} \tag{2.4}$$

The expectation of the estimate of irradiance is:

$$E[\hat{I}(r_k)] = \int_a^b \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k \tag{2.5}$$

As the photons are stratified, for calculating the expectation of the irradiance we do not have to integrate over the whole interval since the position of k^{th} photon can be determined more precisely: The k^{th} nearest photon is situated within a certain subdivision of the interval. In the following section, we model stratified photon mapping with the help of these subdivisions. For this purpose we divide the modelling into two cases, like the first approximation is suggesting - one for even n / even k and odd n / odd k, and one for even n / odd k and odd n / even k. For all models that follow we will deal with the simplified scene, where photons from the negative side are projected on the positive side on the interval (Figure 2.7).

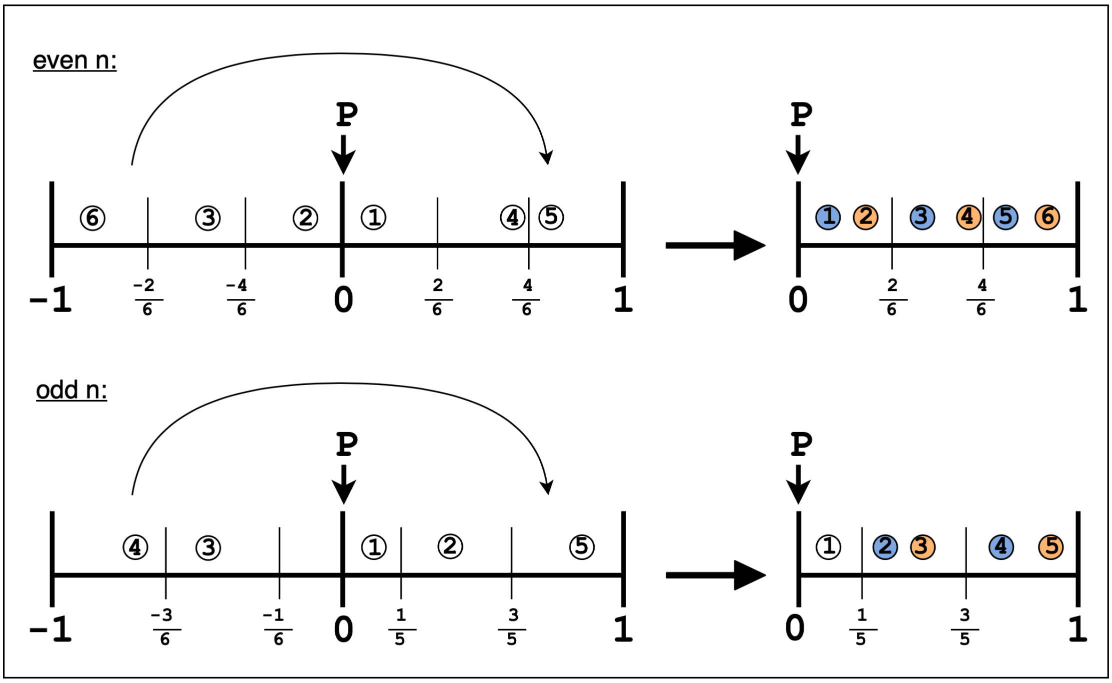


Figure 2.7.: Projection for even n with even k (orange) and odd k (blue); projection for odd n with even k (blue) and odd k (orange).

Modelling for even n / even k and odd n / odd k

When n is even, we are interested in photons with an even k, so the 2^{nd}, 4^{th}, 6^{th}, ... most distant photon to the midpoint P. Generalising the borders of the subdivision $[a, b]$ for photons with even k leads to the result $\left[\frac{k-2}{n}, \frac{k}{n}\right]$. We know for sure that the k^{th} nearest photon is in the subdivision $\left[\frac{k-2}{n}, \frac{k}{n}\right]$ (Figure 2.8).

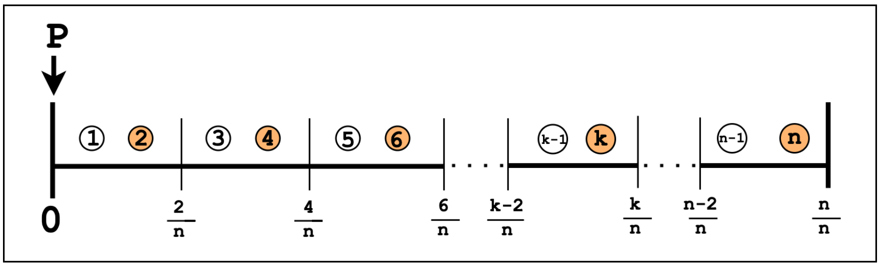


Figure 2.8.: An even number of photons on the projected interval. We are interested in the ones with an even number.

For odd n with $k > 1$ the 3^{rd}, 5^{th}, 7^{th}, ... most distant photon to the midpoint P is relevant. The subdivision for photons with k and $k > 1$ is $\left[\frac{k-2}{n}, \frac{k}{n}\right]$ (Figure 2.9). It is equal to the subdivision for even n and even k. That is why the expected value of the estimate of irradiance is calculated for both cases together. For $k = 1$ there is an exception which will be treated separately later on in this chapter.

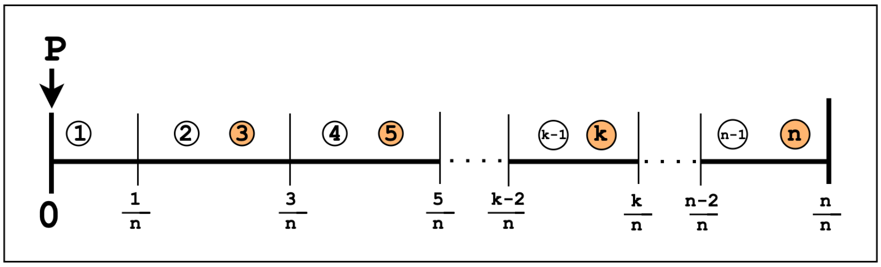


Figure 2.9.: An odd number of photons projected interval. We are interested in the ones with an odd number.

The modelling according to [Gar12] needs the distribution of the $\tilde{k}^{th}$ order statistics (equation (1.3)). $\tilde{n}$ is 2 in our case as we have two photons in the domain that we are integrating over. $\tilde{k}$ is also 2 since we are interested in the photon situated farther away from the midpoint. Equation (1.3) simplifies to:

$$f_{X_{(k)}}(x) = 2F(x)f(x) \tag{2.6}$$

The probability density function f for continuous uniform distributions is for $a <= x <= b$ with $[a, b]$ as our domain [Mat20]:

$$f(r_k) = \frac{1}{b-a} = \frac{1}{\frac{k}{n} - \frac{k-2}{n}} = \frac{n}{2} \tag{2.7}$$

The cumulative distribution function F for uniform distributions for $a <= x <= b$ is again with $[a, b]$ as our domain [Mat20]:

$$F(r_k) = \frac{r_k - a}{b - a} = \frac{r_k - \frac{k-2}{n}}{\frac{k}{n} - \frac{k-2}{n}} = r_k \frac{n}{2} - \frac{k-2}{2} \tag{2.8}$$

With all this information it can be calculated the expectation of the estimator of irradiance from equation (2.5) for even n / even k and odd n / odd k with $k > 1$):

$$E[\hat{I}(r_k)] = \int_{\frac{k-2}{n}}^{\frac{k}{n}} \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k = \int_{\frac{k-2}{n}}^{\frac{k}{n}} \hat{I}(r_k) 2F(r_k) f(r_k) dr_k \tag{2.9}$$

$$= \int_{\frac{k-2}{n}}^{\frac{k}{n}} \frac{kI(P)}{r_k n} 2 \left(r_k \frac{n}{2} - \frac{k-2}{2} \right) \frac{n}{2} dr_k$$

$$= \frac{1}{2} I(P) k (2 + (k-2) Log[k-2] - (k-2) Log[k])$$

The result shows that there is an overestimation of the irradiance. This bias will be examined in more details later on in section 3.1.

Modelling for even n / odd k and odd n / even k

We have a look at the opposite cases where n and k have a different parity. When n is even, we are interested respectively in photons with odd k, so the 1^{st}, 3^{rd}, 5^{th}, ... most distant photon to the midpoint P. Again, we generalize the borders of the subdivisions. The k^{th} nearest photon is situated in the subdivision $\left[\frac{k-1}{n}, \frac{k+1}{n}\right]$ (Figure 2.10). For odd n and even k the generalized borders for the subdivisions are the same as in the previous case: The k^{th} nearest photon is located in $\left[\frac{k-1}{n}, \frac{k+1}{n}\right]$ (Figure 2.11).

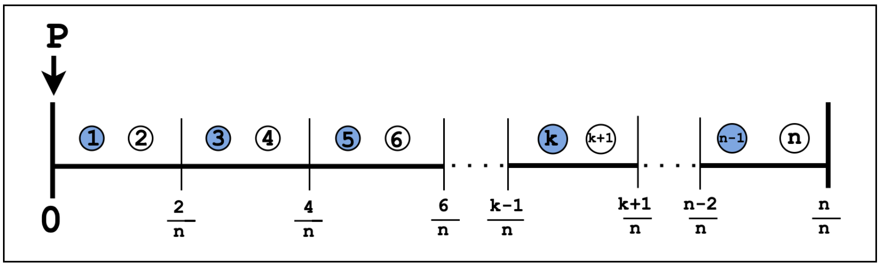


Figure 2.10.: An even number of photons on the projected interval. We are interested in the ones with an odd number.

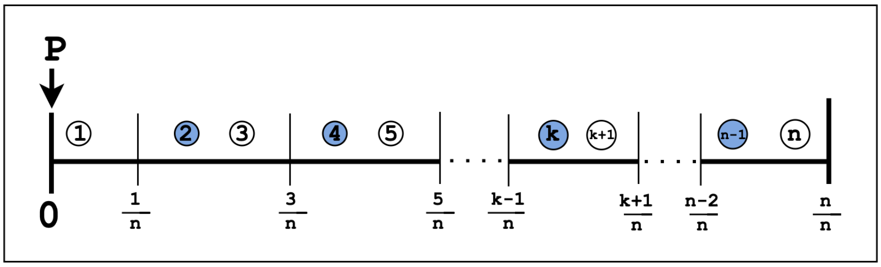


Figure 2.11.: An odd number of photons on the projected interval. We are interested in the ones with an even number.

Next comes the distribution of the $\tilde{k}^{th}$ order statistics (see equation (1.3)). $\tilde{n}$ is 2 in our case as we have two photons in the domain that we are integrating over. $\tilde{k}$ is 1. This represents always that photon in the domain with a smaller distance to the midpoint. For odd k equation 1.3 simplifies to:

$$f_{X_{(k)}}(x) = 2(1 - F(x))f(x) \tag{2.10}$$

The probability density function f for continuous uniform distributions is the same as before for $a <= x <= b$, with $[a, b]$ as our domain [Mat20]:

$$f(r_k) = \frac{1}{b-a} = \frac{1}{\frac{k+1}{n} - \frac{k-1}{n}} = \frac{n}{2} \tag{2.11}$$

The cumulative distribution function F for uniform distributions is for $a <= x <= b$, again with $[a, b]$ as our domain [Mat20]:

$$F(r_k) = \frac{r_k - a}{b - a} = \frac{r_k - \frac{k-1}{n}}{\frac{k+1}{n} - \frac{k-1}{n}} = r_k\frac{n}{2} - \frac{k-1}{2} \tag{2.12}$$

With all this information it can be calculated the expectation of the estimator of irradiance (equation (2.5)):

$$E[\hat{I}(r_k)] = \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k = \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} \hat{I}(r_k) 2(1 - F(r_k)) f(r_k) dr_k \tag{2.13}$$

$$= \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} \frac{kI(P)}{r_k n} 2\left(1 - \left(r_k\frac{n}{2} - \frac{k-1}{2}\right)\right)\frac{n}{2} dr_k$$

$$= -\frac{1}{2} I(P) k(2 + (k+1)Log[k-1] - (k+1)Log[k+1])$$

The bias from this result will also be investigated more closely in section 3.1.

Special case $k = 1$ and odd n

Let us have a look at figure 2.11. The domain for the first photon is $[0, \frac{1}{n}]$. As for $k = 1$ there is only one photon in the subdivision, $\tilde{n} = 1$ and $\tilde{k} = 1$ when looking at the k^{th} order statistics (equation (1.3)).

The probability density function f for continuous uniform distributions is for $a <= x <= b$ [Mat20]:

$$f(r_1) = \frac{1}{b-a} = \frac{1}{\frac{1}{n} - 0} = n \tag{2.14}$$

The cummulative distribution function F in equation (1.3) cancels out as both $\tilde{n}$ and $\tilde{k}$ are 1.

The expectation of the estimator of irradiance for $k = 1$ is therefore:

$$\begin{aligned} E[\hat{I}(r_1)] &= \int_0^{\frac{1}{n}} \hat{I}(r_1) f_{X_{(1)}}(r_1) dr_1 = \int_0^{\frac{1}{n}} \hat{I}(r_1) f(r_k) dr_1 \\ &= \int_0^{\frac{1}{n}} \frac{1 I(P)}{r_1 n} n dr_1 = \int_0^{\frac{1}{n}} \frac{I(P)}{r_1} dr_1 \end{aligned} \tag{2.15}$$

This integral does not converge. Hence, photon mapping should only be used for $k > 1$.

2.2. Surface illumination with stratified photon mapping

The scene for the one-dimensional setting was a line. In order to simulate surface illumination, we need to add a second dimension the scene to be able to simulate surfaces. Whereas the one-dimensional setup was a simplification for introducing the concepts, surface illumination is the approach that is usually used for applications in computer graphics. In [Gar12] photon mapping with random samples was already examined in 2D on a circle-shaped scene. In the following sections we will also use a circle to simulate stratified surface illumination.

2.2.1. Two-dimensional basic stratification

For a fast and easy model of two-dimensional stratified photon mapping we take a circle-shaped scene. The stratification of the light source creates a secondary stratification on the surface that divides the scene in n equally sized areas. All areas have size π, beginning from one in the middle of the scene until the border (see Figure 2.12). In every area is situated one photon. Its position within the area is random. The query point P is the midpoint of the circle. We are interested in the k nearest photons around P within a distance r_k.

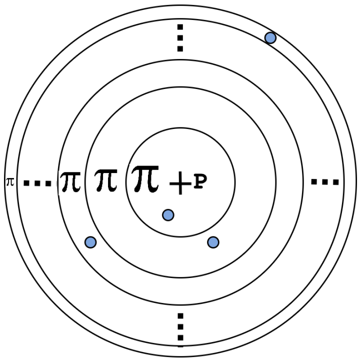


Figure 2.12.: The scene with n photons, each in his own circular subdivision.

Likewise it was done for the one-dimensional models we simplify the scene by projecting the photons on a line. This results in an interval $[0, \sqrt{n}]$ with n subdivisions that contain one photon each (see Figure 2.13).

The irradiance I is constant for all points of the scene. $I(P)$ is the irradiance at query point P. To get the flux ϕ of one photon, $I(P)$ is multiplied by the area of the scene, which n times π, and then divided among the n photons in the scene:

$$\phi = \frac{n\pi I(P)}{n} = \pi I(P) \tag{2.16}$$

The estimate of irradiance $\hat{I}(r_k)$ at P is calculated by taking the flux of the k nearest photons to P and dividing this flux by the area up to the k^{th} nearest photon.

$$\hat{I}(r_k) = \frac{k\phi}{\pi r_k^2} \overset{\text{with (2.16)}}{=} \frac{k\pi I(P)}{\pi r_k^2} = \frac{kI(P)}{r_k^2} \tag{2.17}$$

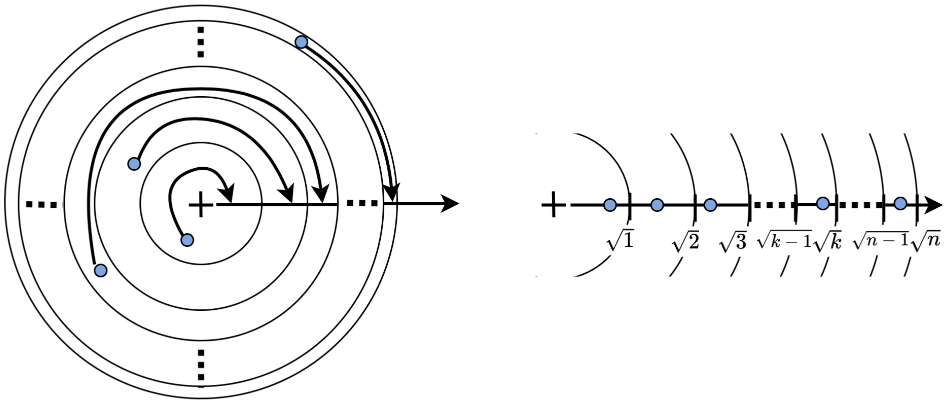


Figure 2.13.: Projecting all photons on a line. In this way, they are ordered by their distance to P.

The expectation of the estimate of irradiance for an interval $[a, b]$ with $f_{X_{(k)}}$ as the $\tilde{k}^{th}$ order statistic is:

$$E[\hat{I}(r_k)] = \int_a^b \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k \tag{2.18}$$

Since the photons are stratified the position of each photon can be determined quite precisely. As it can be seen in figure 2.13, the k^{th} nearest photon is in the subdivision $[\sqrt{k-1}, \sqrt{k}]$.

For the $\tilde{k}^{th}$ order statistic, $\tilde{n} = 1$ as in each section is only one photon. $\tilde{k} = 1$ for the same reason.

$$\begin{aligned} f_{X_{(k)}}(r_k) &= \frac{\tilde{n}!}{(\tilde{k}-1)!(\tilde{n}-\tilde{k})!} F^{\tilde{k}-1}(r_k)(1 - F(r_k))^{\tilde{n}-\tilde{k}} f(r_k) \\ &= \frac{1!}{(1-1)!(1-1)!} F^{1-1}(r_k)(1 - F(r_k))^{1-1} f(r_k) \\ &= f(r_k) \end{aligned} \tag{2.19}$$

The probability of finding photon k in the interval $[\sqrt{k-1}, \sqrt{k}]$ is proportional to the length of the circumference of a circle with radius r_k, so $2\pi r_k$. Integrating this probability between the interval borders gives the area of this circle, which is π. The probability density function is the probability of finding the k^{th} photon by the area of the circle:

$$f(r_k) = \frac{2\pi r_k}{\pi} = 2r_k \tag{2.20}$$

The expectation of the estimate of irradiance with the probability function from equation 2.20 is:

$$E[\hat{I}(r_k)] = \int_{\sqrt{k-1}}^{\sqrt{k}} \frac{kI(P)}{r_k^2} 2r_k dr_k = kI(P)(-Log[k-1] + Log[k]) = kI(P) Log\left(\frac{k}{k-1}\right) \tag{2.21}$$

$I(P)$ is overestimated by $kLog(\frac{k}{k-1})$. In section 3.2 we will have a look at this bias.

2.2.2. Two-dimensional stratified sampling with order statistics

In a second model for surface illumination we try to distribute the photons a bit more uniformly. Again, we have n pieces with area π. The most inner circle is just one piece, the second inner circle is divided in two pieces, the third inner circle in three and so on (Figure 2.14). n must be a triangular number. The radius of the scene is $\sqrt{n}$.

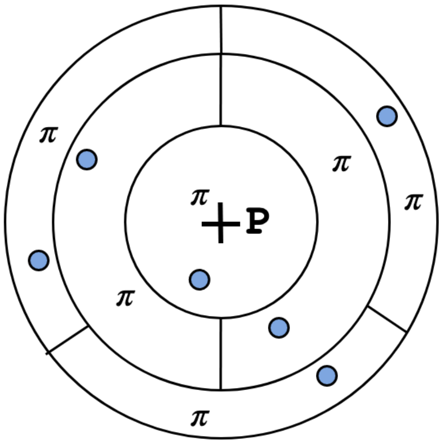


Figure 2.14.: The scene is a circle subdivided in smaller circle. Each circle is then subdivided according to its position in the scene.

When projecting the photons on a line as it was done before, it results in subsections where the first subsection contains one photon, the second subsection two, the third subsection three and so on (Figure 2.15).

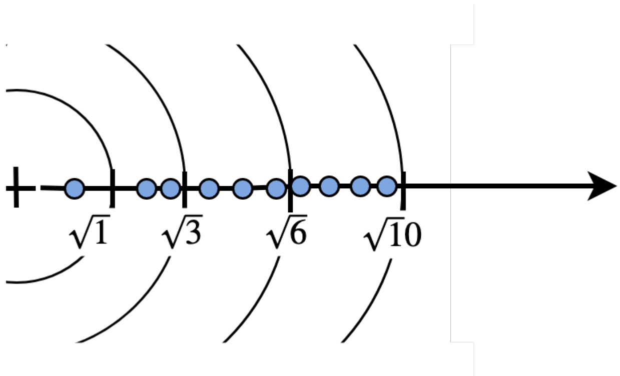


Figure 2.15.: Projection of the photons on a line.

The estimate of irradiance is same as in the first model:

$$\hat{I}(r_k) = \frac{kI(P)}{r_k^2} \tag{2.22}$$

The expectation of estimate of irradiance is:

$$E[\hat{I}(r_k)] = \int_a^b \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k \tag{2.23}$$

The $\tilde{k}^{th}$ order statistics depend on the interval where the photon is situated and at which position within the interval the photon is. For example when looking at the 4^{th} nearest photon, $\tilde{n}$ is 3 as there are three photons in the interval and $\tilde{k}$ is 1 as the 4^{th} nearest photon is the first photon in this interval.

$$f_{X_{(\tilde{k})}}(r_k) = \frac{\tilde{n}!}{(\tilde{k}-1)!(\tilde{n}-\tilde{k})!} F^{\tilde{k}-1}(r_k)(1-F(r_k))^{\tilde{n}-\tilde{k}} f(r_k) \tag{2.24}$$

The probability density function for uniform distributions is [Mat20]:

$$f(r_k) = \frac{1}{b-a} \tag{2.25}$$

And the cumulative distribution function for uniform distributions is [Mat20]:

$$f(r_k) = \frac{r_k - a}{b-a} \tag{2.26}$$

All together, the expectation of the estimate of irradiance is:

$$E[\hat{I}(r_k)] = \int_a^b \frac{kI(P)}{r_k^2} \frac{\tilde{n}!}{(\tilde{k}-1)!(\tilde{n}-\tilde{k})!} \left(\frac{r_k - a}{b-a}\right)^{\tilde{k}-1} \left(1 - \frac{r_k - a}{b-a}\right)^{\tilde{n}-\tilde{k}} \frac{1}{b-a} dr_k \tag{2.27}$$

There is no empirical solution for this integral. In section 3.2 it will be shown graphically that the integral converges and that there is an overestimation bias also for this case.

2.3. Volumetric effects with stratified photon mapping

In the previous section we saw surface illumination with a two-dimensional scene. Volumetric effects appear when light is not travelling in a vacuum. They make it possible for the viewer to see beams of the light source shining through the scene. To simulate these volumetric effects it is needed a scene with three dimensions. In [Gar12] it was used a sphere to examine these effects for random photon mapping. In the following we will use two different methods to subdivide the sphere in order to simulate stratification, like it was done also for two-dimensional stratified photon mapping.

2.3.1. Three-dimensional basic stratification

This approach has a stratified light source that sends out photons on a sphere-shaped scene. The resulting secondary stratification divides the sphere from the inside to the outside into n equally sized, sphere-shaped pieces (Figure 2.16. All pieces have the volume $\frac{4}{3}\pi$ and in each piece is located one photon. The query point P is the midpoint of the sphere. The photon mapping query considers the k nearest photons with respect to P.

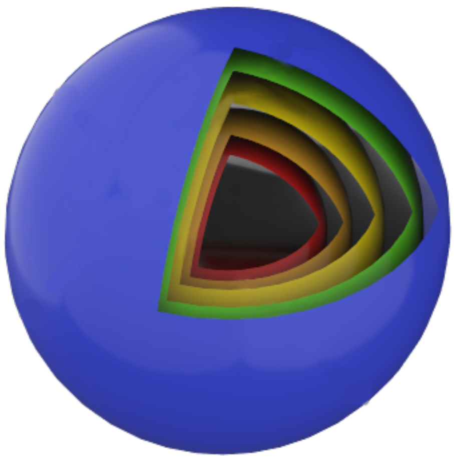


Figure 2.16.: The scene is a sphere that is subdivided in smaller spheres. In each subdivision-sphere is one photon.

In the same way as in the one-dimensional and the two-dimensional case the scene is simplified by projecting the photons on a line and in this way ordering them by their distance to the query point, see figure 2.17.

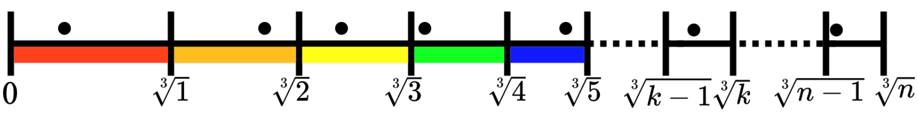


Figure 2.17.: The photons from the sphere are projected on a line.

With a difference to one- and two-dimensional photon mapping, for volumetric effects we are not interested in the irradiance but in the power density at the query point. The power density PD is the power W divided by the volume V [Gar12].

$$PD = \frac{W}{V} \tag{2.28}$$

The volume V of one subdivision is declared as $\frac{4}{3}\pi$. To verify that this is also true for the k^{th} subdivision, we integrate the surface area of the sphere with radius r_k between the interval borders of the k^{th} nearest photon, see figure 2.17.

$$V = \int_{\sqrt[3]{k-1}}^{\sqrt[3]{k}} 4\pi {r_k}^2 dr_k = \frac{4}{3}\pi \tag{2.29}$$

The energy that one photon carries, its flux ϕ, is the total power divided by the number of photons. The total power can be substituted by combining equation (2.28) and (2.29.

$$\phi = \frac{W}{n} = \frac{3PD}{4\pi n} \tag{2.30}$$

The power density estimate $\widehat{PD}$ takes the flux of the k nearest photons and divides it until the volume until the k^{th} nearest photon.

$$\widehat{PD}(r_k) = \frac{k\phi}{\frac{4}{3}\pi {r_k}^3} = \frac{kPD}{{r_k}^3} \tag{2.31}$$

The expectation of the power density estimate for an interval $[a, b]$ with $f_{X_{(k)}}$ as the k^{th} order statistic is:

$$E[\widehat{PD}(r_k)] = \int_a^b \hat{I}(r_k) f_{X_{(k)}}(r_k) dr_k \tag{2.32}$$

Since the photons are stratified the position of each photon can be determined quite precisely. As it can be seen in figure 2.17, the k^{th} nearest photon is in the subdivision $[\sqrt[3]{k-1}, \sqrt[3]{k}]$.

For the $\tilde{k}^{th}$ order statistic, $\tilde{n} = 1$ as in each section is only one photon. $\tilde{k} = 1$ for the same reason.

$$f_{X_{(k)}}(r_k) = f(r_k) \tag{2.33}$$

The probability of finding photon k in the interval $[\sqrt[3]{k-1}, \sqrt[3]{k}]$ is proportional to the length of the surface area of a sphere with radius r_k, so $4\pi r_k^2$. Integrating this probability between the interval borders gives the volume of this sphere, which is $\frac{4}{3}\pi$. The probability density function is the probability of finding the k^{th} photon divided by the area of this sphere:

$$f(r_k) = \frac{4\pi {r_k}^2}{\frac{4}{3}\pi} = 3{r_k}^2 \tag{2.34}$$

Now we can calculate the expectation of the power density estimate:

$$E[\hat{PD}(r_k)] = \int_{\sqrt[3]{k-1}}^{\sqrt[3]{k}} \hat{PD}(r_k) f_{X_{(k)}}(r_k) dr_k = \int_{\sqrt[3]{k-1}}^{\sqrt[3]{k}} \frac{kPD}{{r_k}^3} 3{r_k}^2 dr_k \tag{2.35}$$

$$= 3kPD \left(\frac{Log(k)}{3} - Log(\sqrt[3]{k-1}) \right) = kPDLog\left(\frac{k}{k-1} \right) \tag{2.36}$$

Likewise it was already seen for the one-dimensional case and for surface illumination, also this result shows an overestimation of the power density.

2.3.2. Three-dimensional stratified sampling with order statistics

A different stratification of the light source area causes a different secondary stratification on the scene. For a better photon distribution, all the spheres are subdivided. The most inner sphere has no subdivision, the second inner sphere is divided in two parts, the third inner in three and so on (Figure 2.18) All subdivisions have the size $\frac{4}{3}\pi$ and contain one photon each. The photon mapping query point P is the midpoint of the scene.

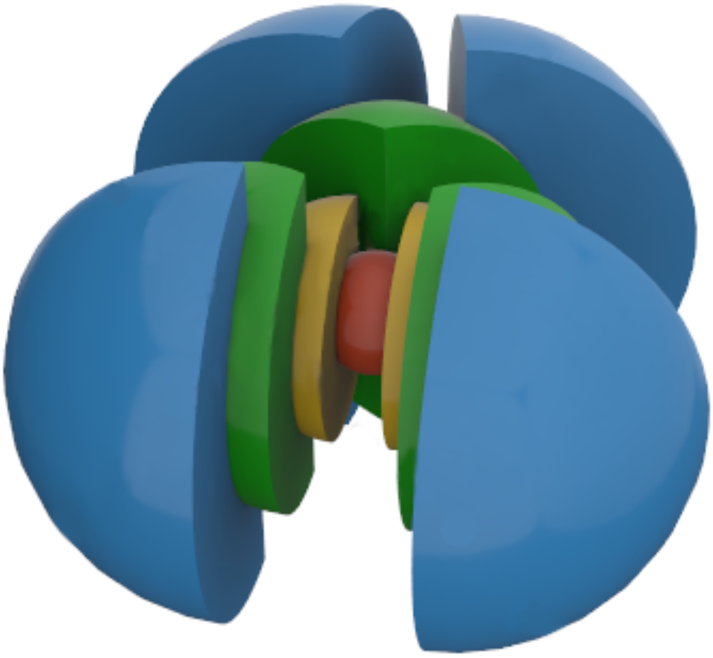


Figure 2.18.: The scene with the spheres that are subdivided in pieces. In each subdivision is one photon

The photons are ordered by their distance to the query point P. This can again be seen as a projection of the photons on a line. According to the subdivisions there is one photon in the first part of the line, there are two in the second part of the line, three in the third part and so on (Figure 2.19.

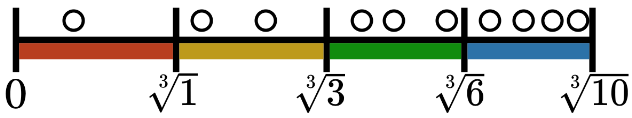


Figure 2.19.: Projecting all photon from the scene on a line.

The power density estimate $\hat{PD}$ is the same as in the model before:

$$\widehat{PD}(r_k) = \frac{k\phi}{\frac{4}{3}\pi {r_k}^3} = \frac{kPD}{{r_k}^3} \tag{2.37}$$

The $\tilde{k}^{th}$ order statistics depend on the subdivision where the photon is situated and at which position within the subdivision the photon is. For example when looking at the 4^{th} nearest photon, $\tilde{n}$ is 3 as there are three photons in the subdivision and $\tilde{k}$ is 1 as the 4^{th} nearest photon is the first photon in this subdivision.

$$f_{X_{(k)}}(r_k) = \frac{\tilde{n}!}{(\tilde{k}-1)!(\tilde{n}-\tilde{k})!} F^{\tilde{k}-1}(r_k)(1-F(r_k))^{\tilde{n}-\tilde{k}} f(r_k) \tag{2.38}$$

The probability density function for uniform distributions is [Mat20]:

$$f(r_k) = \frac{1}{b-a} \tag{2.39}$$

And the cumulative distribution function for uniform distributions is [Mat20]:

$$f(r_k) = \frac{r_k - a}{b-a} \tag{2.40}$$

All in all, the expectation of the power density estimate can be calculated with the following equation.

$$E[\widehat{PD}(r_k)] = \int_a^b \frac{kI(P)}{r_k^2} \frac{\tilde{n}!}{(\tilde{k}-1)!(\tilde{n}-\tilde{k})!} \left(\frac{r_k-a}{b-a}\right)^{\tilde{k}-1} \left(1-\frac{r_k-a}{b-a}\right)^{\tilde{n}-\tilde{k}} \frac{1}{b-a} dr_k \tag{2.41}$$

There is no empirical solution for this integral. In section 3.3 it will be shown graphically that the integral converges and that there is an overestimation bias also for this case.

3. Experimental study of stratified photon mapping

The results from the previous sections are examined in a study of the graphs of the results. For each dimension we implemented the setting in a C-program and calculated the experimental results. The graphs compare the results from the empirical study with the results from these experiments.

3.1. One-dimensional implementation

In the one-dimensional implementation we were differentiating between even n / even k, odd n / odd k and even n / odd k, odd n / even k. The result of the expected value of the estimate of irradiance for n and k with the same parity was:

$$E[\hat{I}(r_k)] = \frac{1}{2}I(P)k(2 + (k-2)Log[k-2] - (k-2)Log[k]) \tag{3.1}$$

Instead, for n and k with a different parity:

$$E[\hat{I}(r_k)] = -\frac{1}{2}I(P)k(2 + (k+1)Log[k-1] - (k+1)Log[k+1]) \tag{3.2}$$

For a plot, we set the irradiance $I(P)$ to one and calculate the relative error by subtracting 1 from each data point.

When comparing the empirical modelling of the 1D case according to [Gar12] with the calculation of the photon mapping irradiance by implementation of the scene in a C program, choosing k again within the interval from 2 to 200, it can be seen that the resulting graphs match. We made two experiments, one with even n ($n = 100\,000$) and one with odd n ($n = 100\,001$).

Figure 3.1 shows a comparison of the empirical and experimental data points only for even n and figure 3.2 shows the comparison only for odd n. In both cases the empirical and experimental points match exactly, this means both models are correct. The relative error shows a clear overestimation bias for all kinds of n and k.

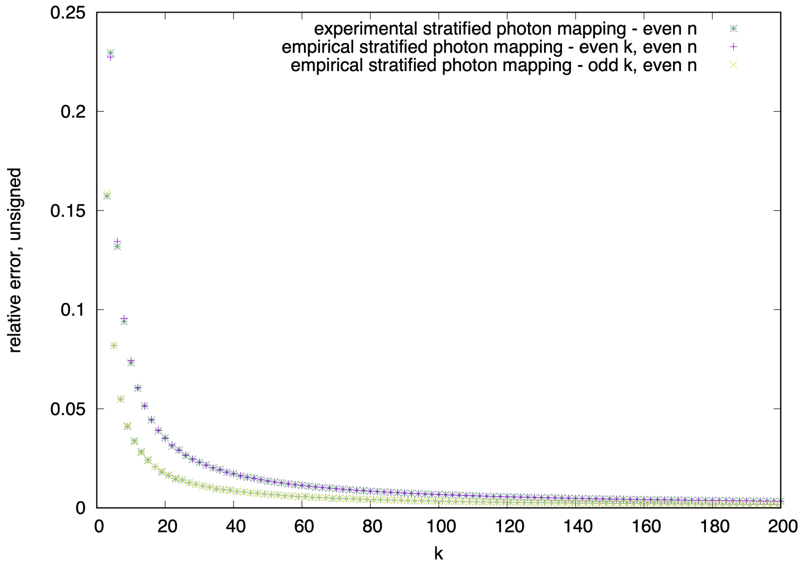


Figure 3.1.: Plot of the relative error of the irradiance from simulating the photon mapping algorithm with stratified samples in a C program with $n = 100\,000$ (even) compared to the relative error of the modelling of the 1D case with stratified samples and even n.

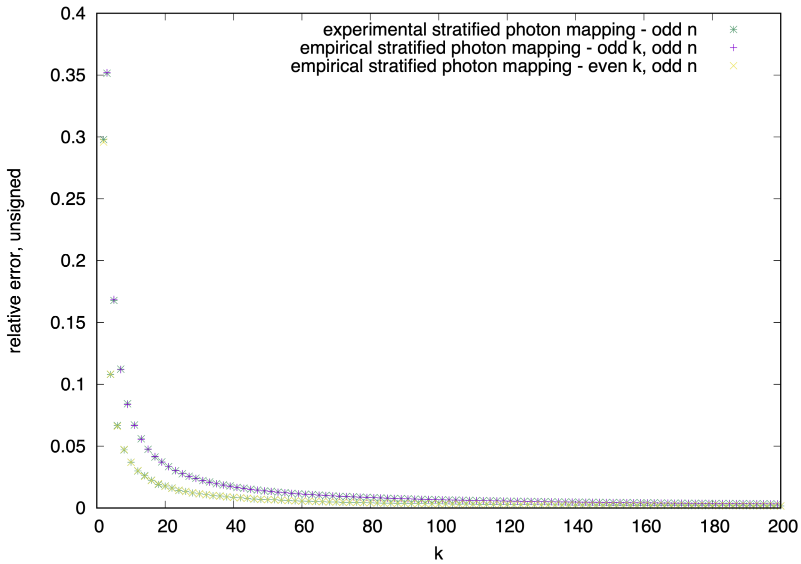


Figure 3.2.: Plot of the relative error of the irradiance from simulating the photon mapping algorithm with stratified samples in a C program with $n = 100\,001$ (odd) compared to the relative error of the modelling of the 1D case with stratified samples and odd n.

3.2. Surface illumination

For surface illumination, we investigated the basic model and the model for stratified sampling with order statistics. Let us first look at the result of basic stratification:

$$E[\hat{I}(r_k)] = kI(P)Log\left(\frac{k}{k-1}\right) \tag{3.3}$$

For showing the relative error, $I(P)$ is set to one and it is subtracted 1 from each data point. In figure 3.3, the empirical relative error is compared with the relative error calculated in in a C-program implementation of the scene. We set n to 1000, chose k within the interval from 2 to 200 and repeated the experiment a thousand times.

Also for two-dimensional stratified sampling the experimental relative error is compared with the relative error calculated in in a C program implementation of the scene. As the integral from equation 2.27 has no solution, we set n for both the C program and the integral n to 1000s and chose k within the interval from 2 to 100. The C program was run with a thousand iterations. The results of the data points of both the empirical and experimental approach are shown in figure 3.4.

For both models of surface illumination, points of the empirical study and the experiment are fitting quite well together. This means that the models should be correct. One has to notice the overestimation bias of the irradiance in both models, though.

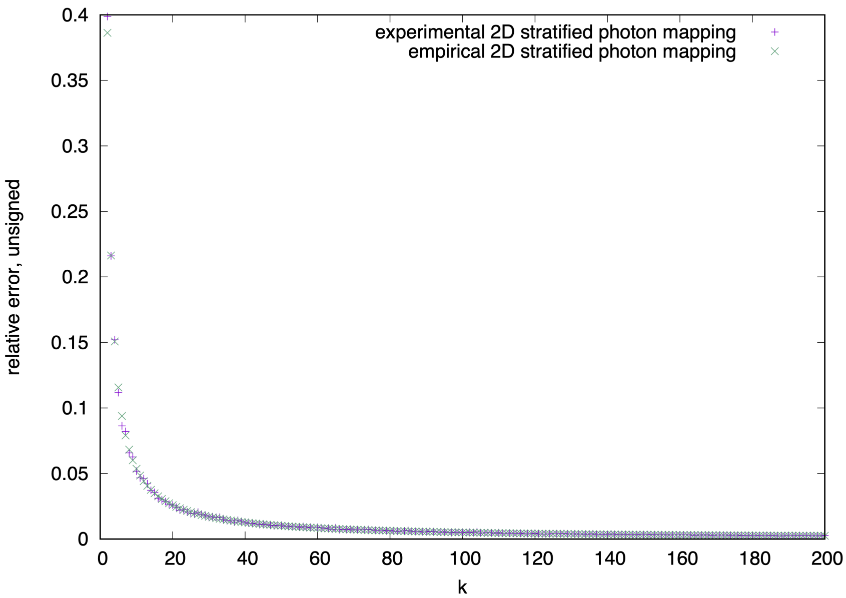


Figure 3.3.: Plot of the relative error for 2D stratified photon mapping, basic stratification. Comparison of the empirical result with the experimental implementation.

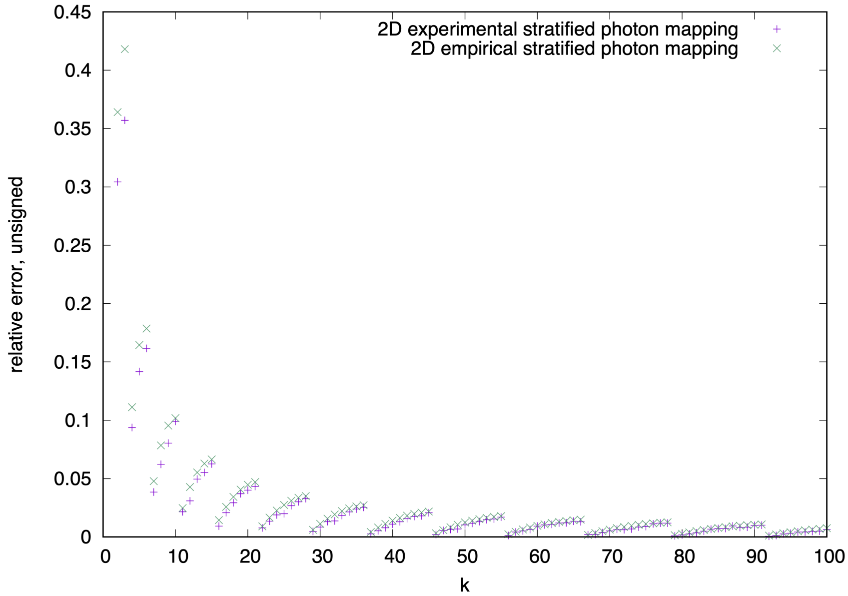


Figure 3.4.: Plot of relative error for 2D stratified photon mapping, stratification with order statistics. Comparison of the empirical result with the experimental implementation.

3.3. Volumetric effects

The result for the expectation of the power density estimator with basic stratification was:

$$E[\hat{PD}(r_k)] = kPDLog\left(\frac{k}{k-1}\right) \tag{3.4}$$

One might notice that this result is the same as the one of surface illumination with basic stratification. This is because the scenes that we used are structured in the same way.

As it was done for one-dimensional and two-dimensional stratified photon mapping, the scene was implemented in a C program and the results are compared with the ones from the empirical model ($n = 1000$, $k = 200$ and number of iterations $= 1000$) in figure 3.5.

The integral form equation (2.41) has no solution, this is why we computed the data points of its solution for $n = 1000$ and k from 2 to 200. These results are compared with the result of the experimental implementation with the same values in figure 3.6.

Also for the simulations of volumetric effects an overestimation of the power density is clearly visible. In the next chapter we will deal with two different approaches that try to reduce this bias.

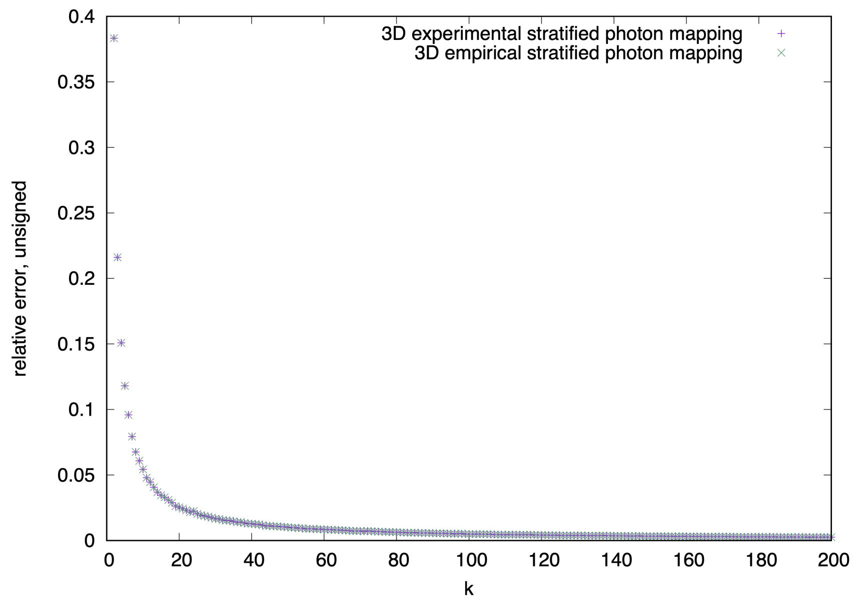


Figure 3.5.: Plot of the relative error for 3D stratified photon mapping, basic stratification. Comparison of the empirical result with the experimental implementation.

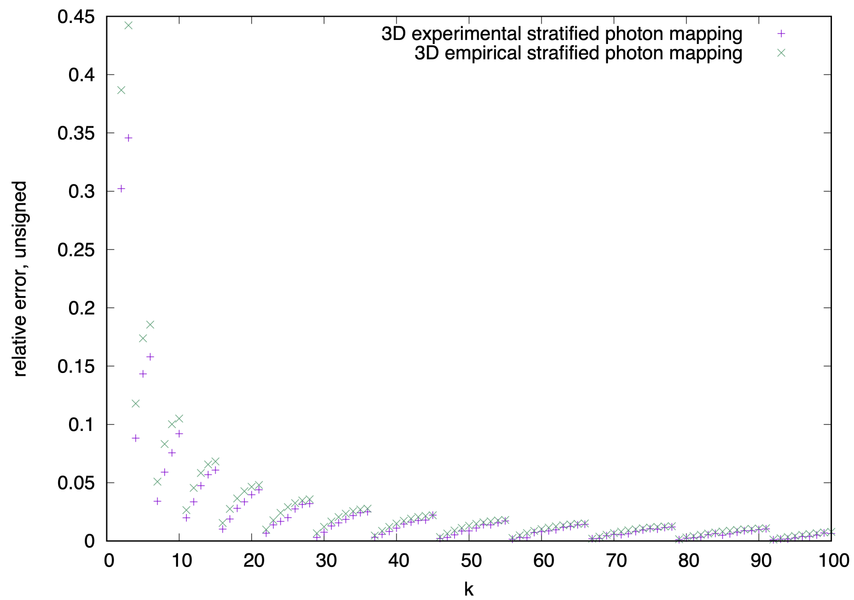


Figure 3.6.: Plot of relative error for 3D stratified photon mapping, stratification with order statistics.Comparison of the empirical result with the experimental implementation.

4. Stratified photon mapping with different estimates

As the previous chapter has shown, all dimensions of stratified photon mapping have an overestimation bias. In history there were provided multiple approaches to remove the overestimation bias of the photon mapping algorithm. In this section we focus on two approaches that have been presented in [Gar12] and [Jen02]. We will combine them with one-dimensional stratified photon mapping, stratified surface illumination and stratified volumetric effects.

4.1. Discarding one impact in the photon mapping query

Like it was already mentioned in the introduction, in [Gar12] the bias of classical photon mapping was removed by changing the algorithm in the following way: Only the flux of the $k-1$ photons is added to calculate the irradiance but still the area r_k until the k^{th} nearest photon is considered.

For one-dimensional photon mapping the estimate of irradiance changes in the following way when discarding the k^{th} nearest photon:

$$\hat{I}(r_k) = \frac{(k-1)I(P)}{r_k n} \tag{4.1}$$

Also for surface illumination we have a new estimate when considering only the $k^{th}-1$ nearest photons:

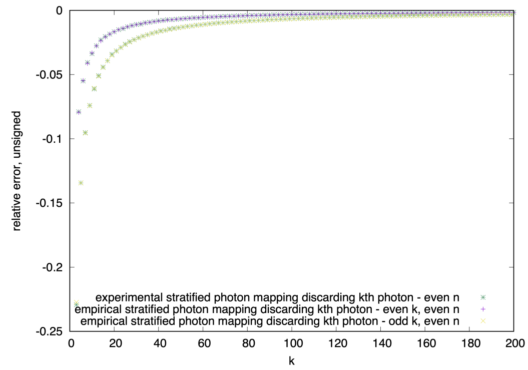
$$\hat{I}(r_k) = \frac{(k-1)I(P)}{r_k^2} \tag{4.2}$$

For volumetric effects the power density estimate when discarding one photon is:

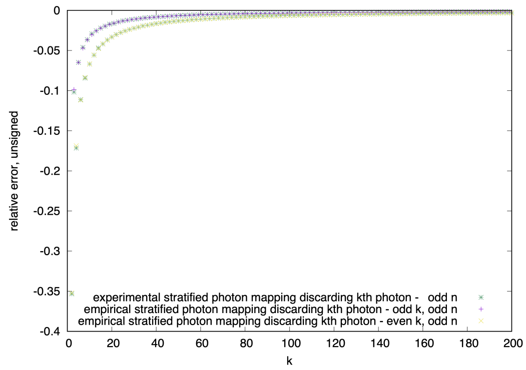
$$\widehat{PD}(r_k) = \frac{(k-1)PD}{{r_k}^3} \tag{4.3}$$

With these new estimates the expected value of the irradiance can be calculated with the same procedure as it was done in chapter 2. The relative errors for the data points for k from 2 to 200 are compared with an experimental implementation of the algorithm discarding the k^{th} nearest photon, throwing 100 000 photons on the scene and repeating the experiment ten thousand times. The results for the one dimensional implementation for even n can be seen in figure 4.1a and the results for odd n in figure 4.1b. Both graphs show that the estimate of [Gar12] produces a small underestimation bias for stratified photon mapping.

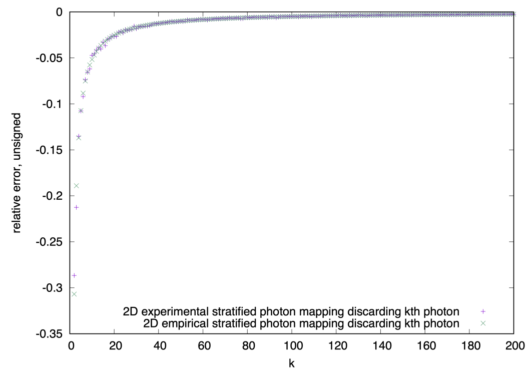
Also for surface illumination and volumetric effects there is an underestimation bias visible. The figures 4.1c and 4.1e show the results for basic stratification. The figures 4.1d and 4.1f instead show the results for the second model with order statistics. For both models there was done again a comparison graph with an experimental implementation of the scene where the number n of the photons was set to 1000 and the experiment was repeated a thousand times.



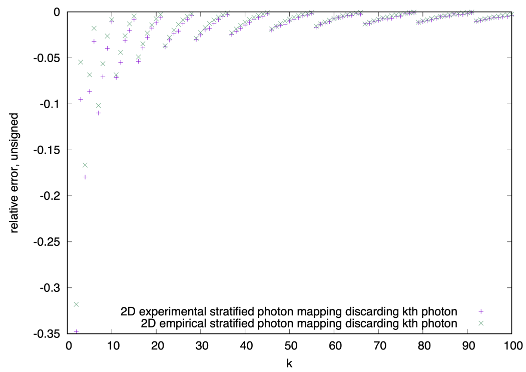
(a) The new estimate discards the k^{th} photon: even n, graphs of relative error.



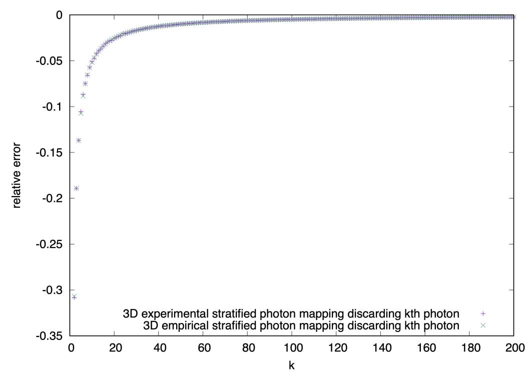
(b) The new estimate discards the k^{th} nearest photon: odd n, graphs of relative error.



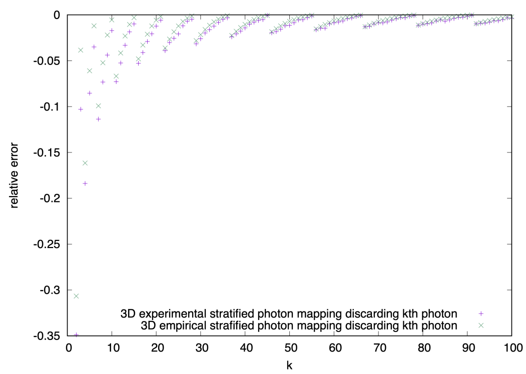
(c) Plot of relative error for 2D stratified photon mapping discarding the k^{th} nearest photon, basic stratification.



(d) Plot of relative error for 2D stratified photon mapping discarding the k^{th} nearest photon, stratification with order statistics.



(e) Plot of relative error for 3D stratified photon mapping, basic stratification, discarding k^{th} nearest photon



(f) Plot of relative error for 3D stratified photon mapping, stratification with order statistics, discarding k^{th} nearest photon

Table 4.1.: Graphs with relative error for an estimate that considers all photons up to the $k-1$ nearest.

4.2. Discarding half impact in the photon mapping query

In [Jen02] it was proposed consider only the flux of the $k-\frac{1}{2}$ nearest photons. This approach was presented in “A Practical Guide to Global Illumination using Photon Mapping”. The idea is to only add half of the power of the k^{th} nearest photon to the irradiance but still consider the whole area r_k up to the k^{th} nearest photon [Jen02].

For one-dimensional photon mapping when discarding half of the k^{th} nearest photon the estimate of irradiance is:

$$\hat{I}(r_k) = \frac{(k-\frac{1}{2})I(P)}{r_k n} \tag{4.4}$$

Also for surface illumination the estimate of irradiance changes:

$$\hat{I}(r_k) = \frac{(k-\frac{1}{2})I(P)}{r_k^2} \tag{4.5}$$

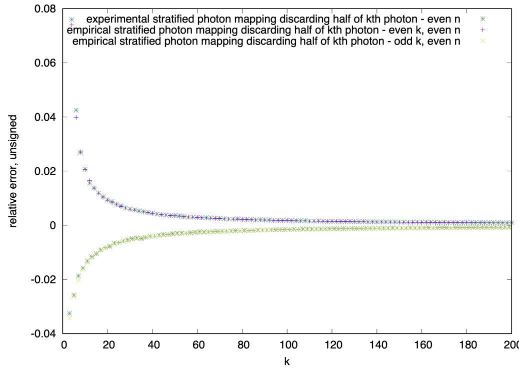
And with volumetric effects the power density estimate that discards half of the k^{th} nearest photon is:

$$\widehat{PD}(r_k) = \frac{(k-\frac{1}{2})PD}{{r_k}^3} \tag{4.6}$$

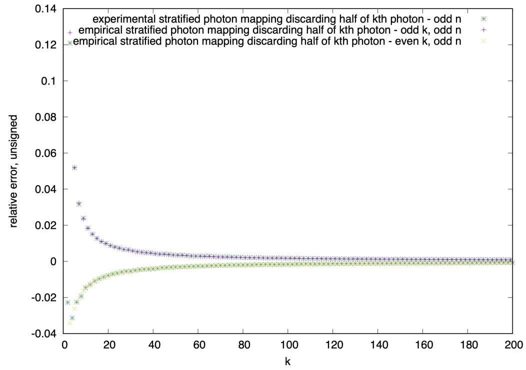
Table 4.2 shows the relative error when discarding half of the k^{th} nearest photon for all setups. The results for the one dimensional implementation for even n can be seen in figure 4.2a and 4.2b. The estimate of [Jen02] reduces the relative error significantly. The figures 4.2c and 4.2d show what happens when discarding half of the k^{th} photon for surface illumination. Also here is a reduction of the bias visible. The same results can be observed for volumetric effects in the figures 4.2e and 4.2f.

In table 4.3 it is presented a comparison of all estimates that we came across for the different models. The graphs from the figures 4.3a, 4.3b, 4.3c, 4.3d, 4.3e and 4.3f show homogeneous results: The original estimate produces an overestimation bias, the estimate from [Gar12] an underestimation bias. The values that are from the study of the estimate from [Jen02] lie between the values of normal stratified photon mapping and stratified photon mapping discarding one photon. Thus, discarding half of the k^{th} nearest photon leads to the smallest relative error and consequently to the smallest bias of all three methods.

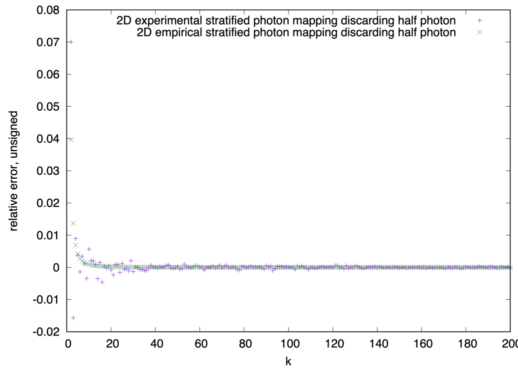
For one-dimensional photon mapping with even n and for two-dimensional basic stratification the relative error is below 8% (figure 4.2a and 4.2c). With two-dimensional and three dimensional stratification with order statistics the relative error is below 20% (figure. For three dimensional basic stratification the relative error goes even below 4% when discarding half impact of the k^{th} nearest photon. The method presented in [Jen02] is helpful when the bias of the photon mapping algorithm should be preferably small. To entirely remove the bias for stratified photon mapping there is no simple method as it was presented for original photon mapping in [Gar12]. It is possible to multiply the estimate by an appropriate constant to achieve a bias of zero, though.



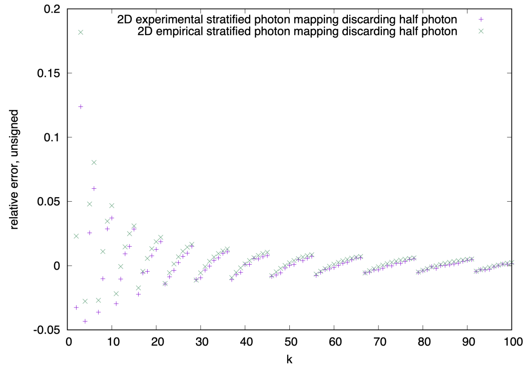
(a) The new estimate discards half of the k^{th} nearest photon: even n, empirical and experimental graph of relative error.



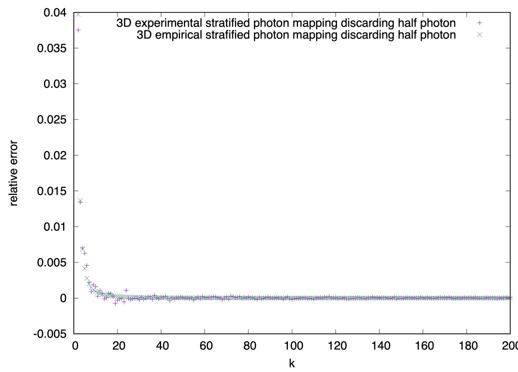
(b) The new estimate discards half of the k^{th} nearest photon: odd n, empirical and experimental graph of relative error.



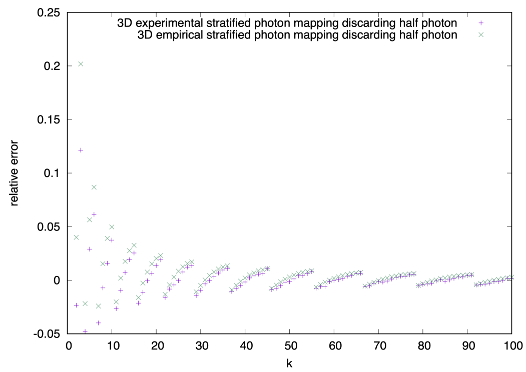
(c) Plot of experimental and empirical 2D stratified photon mapping discarding half of the k^{th} nearest photon, basic stratification.



(d) Plot of experimental and empirical 2D stratified photon mapping discarding half of the k^{th} nearest photon, stratification with order statistics.

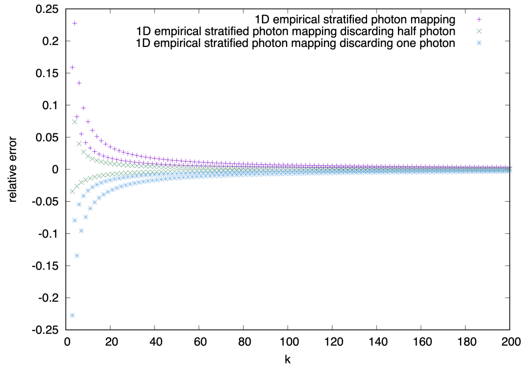


(e) Plot of relative error for 3D empirical and experimental stratified photon mapping, basic stratification, discarding half of the k^{th} nearest photon

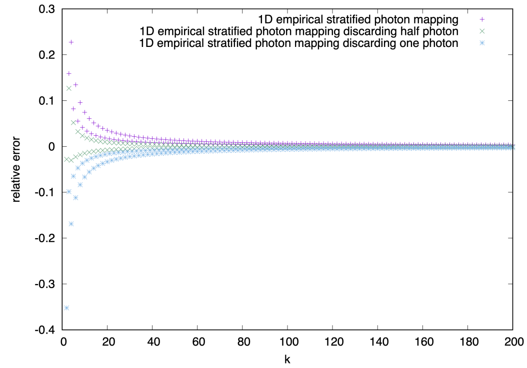


(f) Plot of relative error for 3D empirical and experimental stratified photon mapping, stratification with order statistics, discarding half of the k^{th} nearest photon

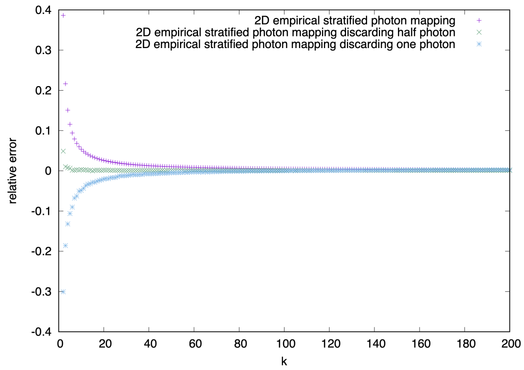
Table 4.2.: Graphs with relative error for an estimate that considers all photons up to the $k - \frac{1}{2}$ nearest.



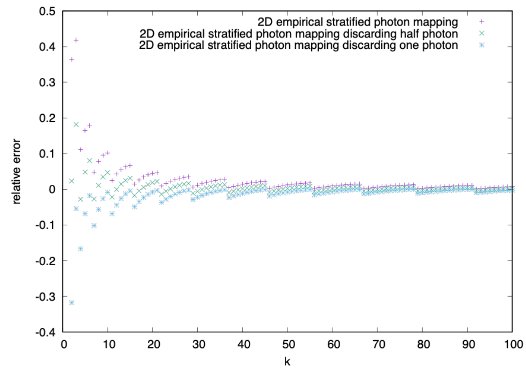
(a) Relative error of all estimates together: one-dimensional photon mapping with even n.



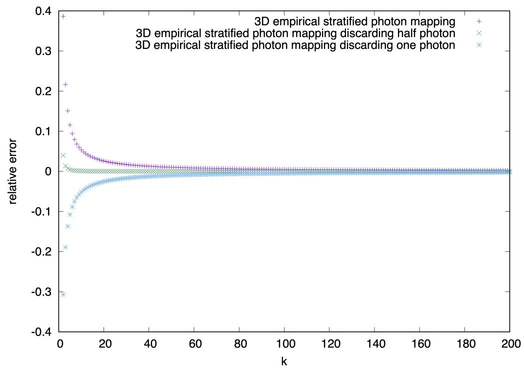
(b) Relative error of all estimates together: one-dimensional photon mapping with odd n.



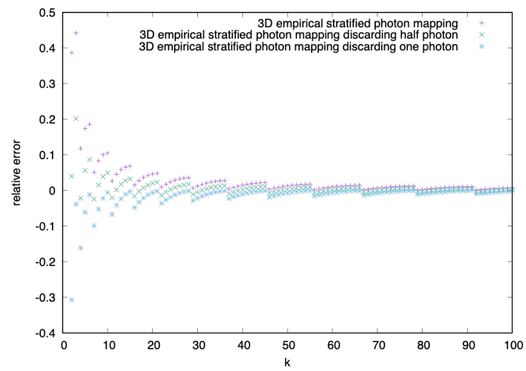
(c) Relative error of all estimates together: surface illumination with basic stratification.



(d) Relative error of all estimates together: surface illumination with stratification and order statistics.



(e) Relative error of all estimates together: volumetric effects with basic stratification.



(f) Relative error of all estimates together: volumetric effects with stratification and order statistics.

Table 4.3.: Graphs with relative error comparing the results of all estimates.

5. Conclusion and Future Work

It is known that biased algorithms can bring to the table interesting properties such as significantly reduced computational time, and they are used very often in practice when the introduced bias is small enough. We analysed the bias introduced by stratified sampling. In general situations stratified sampled is used to reduce bias when sampling a population.

Our results indicate that the overestimation bias is reduced by more than 50% with respect to the original photon mapping. Figure 5.1 shows the significant reduction of the relative error by using stratified sampling. Although this overestimation bias can be removed by discarding the contribution of the k^{th} nearest photon [Gar12], the other desirable properties of stratified sampling may make the use of this algorithm desirable. The 50% remaining bias can either be reduced by discarding half of the k^{th} nearest photon as it was presented in [Jen02] or entirely removed by multiplying with an appropriate constant if necessary; but this step is not as intuitive as the solution for standard photon mapping.

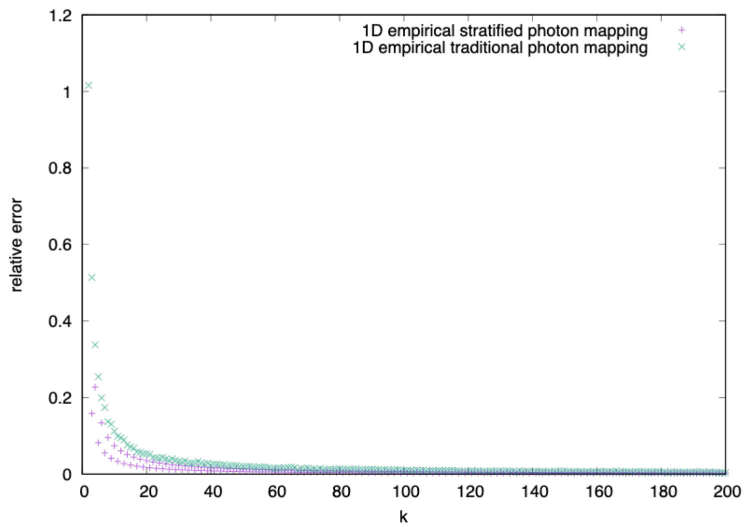


Figure 5.1.: Comparison of relative error from original photon mapping and stratified photon mapping.

5.1. Applications of photon mapping density estimation

The approach to investigate photon mapping that is shown in this work uses the concept of nearest neighbour density estimation. This is an important concept which is also used in many other areas. [FS20] for example shows a clinical research where nearest neighbourhood density estimation is used to cluster patients by their syndrome for heart failures. In [YZ20] on the other hand it is developed a remaining life estimation for lithium ion batteries with the help a k-nearest neighbour regression. Eventually, [YL20] presents an approach for product form design, using density estimation to find out the satisfaction of the customers. This shows that even if photon mapping might not be the most up to date technique for illumination in computer graphics at the moment, the concept behind the photon mapping algorithm has many recent applications.

5.2. Study of variance for stratified photon mapping

As stratification in statistics is known as a method to reduce the variance [Gen05] it might be of interest to study the variance of stratified photon mapping. In [Gan18] is presented the variance of original photon mapping, also with different kernels. The study of variance for stratified photon mapping would thus be a continuation of [Gan18] and of the work presented in this thesis.

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A. Mathematica code

1D random photon mapping

The variables:

- Ir: irradiance at query point
- IrHat: expected value of the estimate of irradiance
- n: total number of photons
- k: number of photons sought in the photon map query
- rk: distance from the query point to the kth nearest photon

```
In[22]:= IrHat = Integrate [(k * Ir) / (rk * n) (n!) / ((k - 1)! (n - k)!) rk ^ (k - 1) (1 - rk) ^ (n - k), {rk, 0, 1}]
```

Out[22]= $\frac{\text{Ir k Gamma}[-1+k]}{\text{Gamma}[k]}$ if Re[k] < 1 + Re[n] && Re[k] > 1

```
In[23]:= FullSimplify [IrHat]
```

Out[23]= $\frac{\text{Ir k}}{-1+k}$ if Re[k] < 1 + Re[n] && Re[k] > 1

1D stratified photon mapping

even n , even k / odd n, odd k:

```
In[34]:= IrHat = Integrate [(k * Ir) / (rk * n) * 2 * (rk * n / 2 - (k - 2) / 2) * (n / 2), {rk, (k - 2) / n, k / n}]
```

Out[34]= $\frac{1}{2}\text{Ir k}\left(2+(-2+k)\,\text{Log}\left[\frac{-2+k}{n}\right]-(-2+k)\,\text{Log}\left[\frac{k}{n}\right]\right)$ if condition

```
In[35]:= Simplify [IrHat , n > k && k > 1]
```

Out[35]= $\frac{1}{2}\text{Ir k}\left(2+(-2+k)\,\text{Log}[-2+k]-(-2+k)\,\text{Log}[k]\right)$ if k > 2 even n , odd k / odd n, even k:

In[28]:=
```
IrHat =
 Integrate [(k * Ir) / (rk * n) * 2 * (1 - (rk * n / 2 - (k - 1) / 2)) * (n / 2), {rk, (k - 1) / n, (k + 1) / n}]
```

Out[28]=
$$-\frac{1}{2} \text{Ir } k \left(1 - k + (1 + k) \text{ Log}\left[\frac{-1 + k}{n}\right]\right) + \frac{1}{2} \text{Ir } k (1 + k) \left(-1 + \text{Log}\left[\frac{1 + k}{n}\right]\right) \quad \text{if} \quad \textit{condition}$$

In[29]:=
```
Simplify [IrHat , n > k && k > 1]
```

Out[29]=
$$-\frac{1}{2} \text{Ir } k (2 + (1 + k) \text{ Log}[-1 + k] - (1 + k) \text{ Log}[1 + k])$$

Discarding kth nearest impact

even n , even k / odd n, odd k:

In[30]:=
```
IrHat = Integrate [((k - 1) * Ir) / (rk * n) * 2 * (rk * n / 2 - (k - 2) / 2) * (n / 2), {rk, (k - 2) / n, k / n}]
```

Out[30]=
$$\frac{1}{2} \text{Ir } (-1 + k) \left(2 + (-2 + k) \text{ Log}\left[\frac{-2 + k}{n}\right] - (-2 + k) \text{ Log}\left[\frac{k}{n}\right]\right) \quad \text{if} \quad \textit{condition}$$

In[31]:=
```
Simplify [IrHat , n > k && k > 1]
```

Out[31]=
$$\frac{1}{2} \text{Ir } (-1 + k) (2 + (-2 + k) \text{ Log}[-2 + k] - (-2 + k) \text{ Log}[k]) \quad \text{if} \quad k > 2$$

even n , odd k / odd n, even k:

In[32]:=
```
IrHat =
 Integrate [((k - 1) * Ir) / (rk * n) * 2 * (1 - (rk * n / 2 - (k - 1) / 2)) * (n / 2), {rk, (k - 1) / n, (k + 1) / n}]
```

Out[32]=
$$-\frac{1}{2} \text{Ir } (-1 + k) \left(1 - k + (1 + k) \text{ Log}\left[\frac{-1 + k}{n}\right]\right) + \frac{1}{2} \text{Ir } (-1 + k) (1 + k) \left(-1 + \text{Log}\left[\frac{1 + k}{n}\right]\right) \quad \text{if} \quad \textit{condition}$$

In[33]:=
```
Simplify [IrHat , n > k && k > 1]
```

Out[33]=
$$-\frac{1}{2} \text{Ir } (-1 + k) (2 + (1 + k) \text{ Log}[-1 + k] - (1 + k) \text{ Log}[1 + k])$$

Discarding half kth nearest impact

even n , even k / odd n, odd k:

In[36]:=
```
IrHat =
 Integrate [((k - (1 / 2)) * Ir) / (rk * n) * 2 * (rk * n / 2 - (k - 2) / 2) * (n / 2), {rk, (k - 2) / n, k / n}]
```

Out[36]= $\frac{1}{4}\,\mathrm{Ir}\,(-1+2\,k)\left(2+(-2+k)\,\mathrm{Log}\left[\frac{-2+k}{n}\right]-(-2+k)\,\mathrm{Log}\left[\frac{k}{n}\right]\right)$ if condition

In[37]:=
```
Simplify [IrHat , n > k && k > 1]
```

Out[37]= $\frac{1}{4}\,\mathrm{Ir}\,(-1+2\,k)\,(2+(-2+k)\,\mathrm{Log}[-2+k]-(-2+k)\,\mathrm{Log}[k])$ if $k > 2$

even n , odd k / odd n, even k:

In[38]:=
```
IrHat = Integrate [
  ((k - (1 / 2)) * Ir) / (rk * n) * 2 * (1 - (rk * n / 2 - (k - 1) / 2)) * (n / 2), {rk, (k - 1) / n, (k + 1) / n}]
```

Out[38]= $-\frac{1}{4}\,\mathrm{Ir}\,(-1+2\,k)\left(2+(1+k)\,\mathrm{Log}\left[\frac{-1+k}{n}\right]-(1+k)\,\mathrm{Log}\left[\frac{1+k}{n}\right]\right)$ if condition

In[39]:=
```
Simplify [IrHat , n > k && k > 1]
```

Out[39]= $-\frac{1}{4}\,\mathrm{Ir}\,(-1+2\,k)\,(2+(1+k)\,\mathrm{Log}[-1+k]-(1+k)\,\mathrm{Log}[1+k])$

2D stratified photon mapping

The variables:

- Ir: irradiance at query point
- IrHat: expected value of the estimate of irradiance
- n: total number of photons
- k: number of photons sought in the photon map query
- rk: distance from the query point to the kth nearest photon

First Model

```
In[45]:= IrHat = Integrate [((k * Ir) / (rk ^ 2) * 2 * rk , {rk , Sqrt [k - 1], Sqrt [k]}]

Out[45]= Ir k (- Log[- 1 + k] + Log[k])  if  condition

In[46]:= Simplify [IrHat , n > k && k > 1]
```

Out[46]= $\text{Ir k Log}\left[\frac{k}{-1+k}\right]$

Discarding kth nearest impact

```
In[48]:= IrHat = Integrate [((k - 1) * Ir) / (rk ^ 2) * 2 * rk , {rk , Sqrt [k - 1], Sqrt [k]}]

Out[48]= Ir (- 1 + k) (- Log[- 1 + k] + Log[k])  if  condition

In[49]:= Simplify [IrHat , n > k && k > 1]
```

Out[49]= $\text{Ir }(-1+k)\text{ Log}\left[\frac{k}{-1+k}\right]$

Discarding half kth nearest impact

```
In[50]:= IrHat = Integrate [((k - (1 / 2)) * Ir) / (rk ^ 2) * 2 * rk , {rk , Sqrt [k - 1], Sqrt [k]}]
```

Out[50]= $\frac{1}{2}\text{ Ir }(-1+2\text{ k})\,(-\text{Log}[-1+\text{k}]+\text{Log}[\text{k}])$ if condition

In[51]:= Simplify [IrHat , n > k && k > 1]

Out[51]= $\frac{1}{2}\,\mathrm{Ir}\,(-1+2\,k)\,\mathrm{Log}\left[\frac{k}{-1+k}\right]$

Second Model

In[1]:= x = Ceiling [(Ceiling [Sqrt [1 + 8 * k]] - 1) / (2)]

Out[1]= $\mathrm{Ceiling}\left[\frac{1}{2}\left(-1+\mathrm{Ceiling}\left[\sqrt{1+8\,k}\right]\right)\right]$

In[2]:= y1 = (x * (x + 1)) / 2

Out[2]= $\frac{1}{2}\,\mathrm{Ceiling}\left[\frac{1}{2}\left(-1+\mathrm{Ceiling}\left[\sqrt{1+8\,k}\right]\right)\right]\left(1+\mathrm{Ceiling}\left[\frac{1}{2}\left(-1+\mathrm{Ceiling}\left[\sqrt{1+8\,k}\right]\right)\right]\right)$

In[3]:= y2 = (x * (x - 1)) / 2

Out[3]= $\frac{1}{2}\left(-1+\mathrm{Ceiling}\left[\frac{1}{2}\left(-1+\mathrm{Ceiling}\left[\sqrt{1+8\,k}\right]\right)\right]\right)\mathrm{Ceiling}\left[\frac{1}{2}\left(-1+\mathrm{Ceiling}\left[\sqrt{1+8\,k}\right]\right)\right]$

In[4]:= IrHat = Table [N[Integrate [(k * Ir) / (rk ^ 2) *
((x)! / (((k - y2) - 1)! * (x - (k - y2))!)) * ((rk - Sqrt [y2]) / (Sqrt [y1] - Sqrt [y2])) ^ ((k - y2) - 1) *
(1 - ((rk - Sqrt [y2]) / (Sqrt [y1] - Sqrt [y2]))) ^ (x - (k - y2)) *
(1 / (Sqrt [y1] - Sqrt [y2])), {rk, Sqrt [y2], Sqrt [y1]}], 5], {k, 2, 100}]

Out[4]= {1.3640 Ir, 1.4181 Ir, 1.1112 Ir, 1.1645 Ir, 1.1786 Ir, 1.0478 Ir, 1.0784 Ir, 1.0955 Ir,
1.1018 Ir, 1.0247 Ir, 1.0428 Ir, 1.0552 Ir, 1.0629 Ir, 1.0665 Ir, 1.0144 Ir,
1.0257 Ir, 1.0344 Ir, 1.0406 Ir, 1.0447 Ir, 1.0470 Ir, 1.0091 Ir, 1.0166 Ir,
1.0227 Ir, 1.0274 Ir, 1.0310 Ir, 1.0335 Ir, 1.0351 Ir, 1.0061 Ir, 1.0113 Ir,
1.0157 Ir, 1.0193 Ir, 1.0222 Ir, 1.0245 Ir, 1.0261 Ir, 1.0272 Ir, 1.0043 Ir,
1.0080 Ir, 1.0112 Ir, 1.0140 Ir, 1.0164 Ir, 1.0183 Ir, 1.0198 Ir, 1.0209 Ir,
1.0217 Ir, 1.0031 Ir, 1.0059 Ir, 1.0083 Ir, 1.0105 Ir, 1.0123 Ir, 1.0139 Ir,
1.0153 Ir, 1.0163 Ir, 1.0172 Ir, 1.0178 Ir, 1.0023 Ir, 1.0044 Ir, 1.0063 Ir,
1.0080 Ir, 1.0095 Ir, 1.0108 Ir, 1.0120 Ir, 1.0129 Ir, 1.0137 Ir, 1.0143 Ir,
1.0148 Ir, 1.0018 Ir, 1.0034 Ir, 1.0049 Ir, 1.0063 Ir, 1.0075 Ir, 1.0086 Ir,
1.0095 Ir, 1.0104 Ir, 1.0111 Ir, 1.0117 Ir, 1.0122 Ir, 1.0125 Ir, 1.0014 Ir,
1.0027 Ir, 1.0039 Ir, 1.0050 Ir, 1.0060 Ir, 1.0069 Ir, 1.0077 Ir, 1.0084 Ir,
1.0091 Ir, 1.0096 Ir, 1.0101 Ir, 1.0105 Ir, 1.0108 Ir, 1.0011 Ir, 1.0022 Ir,
1.0031 Ir, 1.0040 Ir, 1.0049 Ir, 1.0056 Ir, 1.0063 Ir, 1.0069 Ir, 1.0075 Ir}

Discarding kth nearest impact

In[5]:=
```
IrHat = Table[N[Integrate[((k - 1) * Ir) / (rk ^ 2) *
      ((x)! / (((k - y2) - 1)! * (x - (k - y2))!)) * ((rk - Sqrt[y2]) / (Sqrt[y1] - Sqrt[y2])) ^ ((k - y2) - 1) *
      (1 - ((rk - Sqrt[y2]) / (Sqrt[y1] - Sqrt[y2]))) ^ (x - (k - y2)) *
      (1 / (Sqrt[y1] - Sqrt[y2])), {rk, Sqrt[y2], Sqrt[y1]}], 5], {k, 2, 100}]
```

Out[5]=
```
{0.68201 Ir, 0.94538 Ir, 0.83337 Ir, 0.93157 Ir, 0.98213 Ir, 0.89815 Ir, 0.94363 Ir,
 0.97373 Ir, 0.99166 Ir, 0.93158 Ir, 0.95591 Ir, 0.97406 Ir, 0.98695 Ir, 0.99537 Ir,
 0.95098 Ir, 0.96538 Ir, 0.97689 Ir, 0.98583 Ir, 0.99250 Ir, 0.99715 Ir, 0.96320 Ir,
 0.97238 Ir, 0.98005 Ir, 0.98634 Ir, 0.99137 Ir, 0.99526 Ir, 0.99811 Ir, 0.97139 Ir,
 0.97757 Ir, 0.98290 Ir, 0.98743 Ir, 0.99123 Ir, 0.99434 Ir, 0.99680 Ir, 0.99868 Ir,
 0.97712 Ir, 0.98148 Ir, 0.98532 Ir, 0.98867 Ir, 0.99156 Ir, 0.99402 Ir, 0.99607 Ir,
 0.99774 Ir, 0.99904 Ir, 0.98130 Ir, 0.98448 Ir, 0.98733 Ir, 0.98986 Ir, 0.99210 Ir,
 0.99405 Ir, 0.99573 Ir, 0.99716 Ir, 0.99834 Ir, 0.99928 Ir, 0.98443 Ir, 0.98682 Ir,
 0.98899 Ir, 0.99095 Ir, 0.99271 Ir, 0.99427 Ir, 0.99565 Ir, 0.99685 Ir,
 0.99787 Ir, 0.99874 Ir, 0.99944 Ir, 0.98684 Ir, 0.98868 Ir, 0.99037 Ir,
 0.99191 Ir, 0.99331 Ir, 0.99457 Ir, 0.99571 Ir, 0.99672 Ir, 0.99760 Ir,
 0.99837 Ir, 0.99902 Ir, 0.99956 Ir, 0.98873 Ir, 0.99018 Ir, 0.99151 Ir,
 0.99275 Ir, 0.99388 Ir, 0.99491 Ir, 0.99585 Ir, 0.99670 Ir, 0.99746 Ir,
 0.99813 Ir, 0.99872 Ir, 0.99922 Ir, 0.99965 Ir, 0.99025 Ir, 0.99140 Ir,
 0.99248 Ir, 0.99348 Ir, 0.99440 Ir, 0.99526 Ir, 0.99604 Ir, 0.99676 Ir, 0.99741 Ir}
```

Discarding half kth nearest impact

In[6]:=
```
IrHat = Table[N[Integrate[((k - (1 / 2)) * Ir) / (rk ^ 2) *
      ((x)! / (((k - y2) - 1)! * (x - (k - y2))!)) * ((rk - Sqrt[y2]) / (Sqrt[y1] - Sqrt[y2])) ^ ((k - y2) - 1) *
      (1 - ((rk - Sqrt[y2]) / (Sqrt[y1] - Sqrt[y2]))) ^ (x - (k - y2)) *
      (1 / (Sqrt[y1] - Sqrt[y2])), {rk, Sqrt[y2], Sqrt[y1]}], 5], {k, 2, 100}]
```

Out[6]=
```
{1.0230 Ir, 1.1817 Ir, 0.97226 Ir, 1.0480 Ir, 1.0803 Ir, 0.97300 Ir, 1.0110 Ir, 1.0346 Ir,
 1.0468 Ir, 0.97816 Ir, 0.99936 Ir, 1.0146 Ir, 1.0249 Ir, 1.0309 Ir, 0.98268 Ir,
 0.99555 Ir, 1.0056 Ir, 1.0132 Ir, 1.0186 Ir, 1.0221 Ir, 0.98614 Ir, 0.99448 Ir,
 1.0014 Ir, 1.0069 Ir, 1.0112 Ir, 1.0144 Ir, 1.0166 Ir, 0.98873 Ir, 0.99442 Ir,
 0.99928 Ir, 1.0034 Ir, 1.0067 Ir, 1.0094 Ir, 1.0115 Ir, 1.0129 Ir, 0.99070 Ir,
 0.99474 Ir, 0.99828 Ir, 1.0013 Ir, 1.0040 Ir, 1.0061 Ir, 1.0079 Ir, 1.0093 Ir,
 1.0104 Ir, 0.99220 Ir, 0.99518 Ir, 0.99783 Ir, 1.0002 Ir, 1.0022 Ir, 1.0040 Ir,
 1.0055 Ir, 1.0067 Ir, 1.0078 Ir, 1.0085 Ir, 0.99338 Ir, 0.99563 Ir, 0.99767 Ir,
 0.99949 Ir, 1.0011 Ir, 1.0026 Ir, 1.0038 Ir, 1.0049 Ir, 1.0058 Ir, 1.0065 Ir,
 1.0071 Ir, 0.99432 Ir, 0.99606 Ir, 0.99765 Ir, 0.99910 Ir, 1.0004 Ir, 1.0016 Ir,
 1.0026 Ir, 1.0035 Ir, 1.0043 Ir, 1.0050 Ir, 1.0056 Ir, 1.0061 Ir, 0.99507 Ir,
 0.99644 Ir, 0.99771 Ir, 0.99888 Ir, 0.99994 Ir, 1.0009 Ir, 1.0018 Ir, 1.0026 Ir,
 1.0033 Ir, 1.0039 Ir, 1.0044 Ir, 1.0048 Ir, 1.0052 Ir, 0.99569 Ir, 0.99679 Ir,
 0.99781 Ir, 0.99876 Ir, 0.99964 Ir, 1.0004 Ir, 1.0012 Ir, 1.0018 Ir, 1.0024 Ir}
```

3D stratified photon mapping

The variables:

- PD: power density at query point
- PDHat: expected value of the power density estimate
- n: total number of photons
- k: number of photons sought in the photon map query
- rk: distance from the query point to the kth nearest photon

First Model

In[59]:=
```
PDHat = Integrate [(k * PD) / (rk ^ 3) * 3 * rk ^ 2 , {rk , CubeRoot [k - 1], CubeRoot [k]}]
```

Out[59]= $\boxed{3\, k\, PD \left(\frac{Log[k]}{3} - Log\left[\sqrt[3]{-1+k}\right]\right) \quad \text{if} \quad k \geq 1}$

In[60]:=
```
Simplify [PDHat , n > k && k > 1]
```

Out[60]= $k\, PD\, Log\left[\frac{k}{-1+k}\right]$

Discarding kth nearest impact

In[61]:=
```
PDHat = Integrate [((k - 1) * PD) / (rk ^ 3) * 3 * rk ^ 2 , {rk , CubeRoot [k - 1], CubeRoot [k]}]
```

Out[61]= $\boxed{3\, (-1+k)\, PD \left(\frac{Log[k]}{3} - Log\left[\sqrt[3]{-1+k}\right]\right) \quad \text{if} \quad k \geq 1}$

In[62]:=
```
Simplify [PDHat , n > k && k > 1]
```

Out[62]= $(-1+k)\, PD\, Log\left[\frac{k}{-1+k}\right]$

Discarding half kth nearest impact

In[63]:=
```
PDHat = Integrate [((k - (1 / 2)) * PD) / (rk ^ 3) * 3 * rk ^ 2 , {rk , CubeRoot [k - 1], CubeRoot [k]}]
```

Out[63]= $\boxed{\frac{3}{2}\, (-1+2\, k)\, PD \left(\frac{Log[k]}{3} - Log\left[\sqrt[3]{-1+k}\right]\right) \quad \text{if} \quad k \geq 1}$

In[64]:=
```
Simplify [PDHat , n > k && k > 1]
```

Out[64]= $\frac{1}{2}(-1+2\,k)\,\text{PD}\,\text{Log}\left[\frac{k}{-1+k}\right]$

Second Model

In[3]:=
```
x = Ceiling [(Ceiling [Sqrt [1 + 8 * k]] - 1) / (2)]
```

Out[3]= $\text{Ceiling}\left[\frac{1}{2}\left(-1+\text{Ceiling}\left[\sqrt{1+8\,k}\,\right]\right)\right]$

In[4]:=
```
y1 = (x * (x + 1)) / 2
```

Out[4]= $\frac{1}{2}\,\text{Ceiling}\left[\frac{1}{2}\left(-1+\text{Ceiling}\left[\sqrt{1+8\,k}\,\right]\right)\right]\left(1+\text{Ceiling}\left[\frac{1}{2}\left(-1+\text{Ceiling}\left[\sqrt{1+8\,k}\,\right]\right)\right]\right)$

In[67]:=
```
y2 = (x * (x - 1)) / 2
```

Out[67]= $\frac{1}{2}\left(-1+\text{Ceiling}\left[\frac{1}{2}\left(-1+\text{Ceiling}\left[\sqrt{1+8\,k}\,\right]\right)\right]\right)\text{Ceiling}\left[\frac{1}{2}\left(-1+\text{Ceiling}\left[\sqrt{1+8\,k}\,\right]\right)\right]$

In[68]:=
```
PDHat  = Table [N[Integrate [(k * PD) / (rk ^ 3) * ((x)! / (((k - y2) - 1)! * (x - (k - y2))!)) *
        ((rk - CubeRoot [y2]) / (CubeRoot [y1] - CubeRoot [y2])) ^ ((k - y2) - 1) *
        (1 - ((rk - CubeRoot [y2]) / (CubeRoot [y1] - CubeRoot [y2]))) ^ (x - (k - y2)) *
        (1 / (CubeRoot [y1] - CubeRoot [y2])), {rk, CubeRoot [y2], CubeRoot [y1]}], 5], {k, 2, 100}]
```

Out[68]=
```
{1.3867 PD, 1.4422 PD, 1.1179 PD, 1.1738 PD, 1.1856 PD, 1.0509 PD, 1.0832 PD, 1.1002 PD,
 1.1050 PD, 1.0264 PD, 1.0456 PD, 1.0583 PD, 1.0656 PD, 1.0682 PD, 1.0154 PD,
 1.0275 PD, 1.0365 PD, 1.0427 PD, 1.0465 PD, 1.0480 PD, 1.0098 PD, 1.0178 PD,
 1.0241 PD, 1.0290 PD, 1.0325 PD, 1.0347 PD, 1.0358 PD, 1.0066 PD, 1.0121 PD,
 1.0167 PD, 1.0205 PD, 1.0234 PD, 1.0255 PD, 1.0269 PD, 1.0277 PD, 1.0046 PD,
 1.0086 PD, 1.0121 PD, 1.0149 PD, 1.0173 PD, 1.0192 PD, 1.0206 PD, 1.0215 PD,
 1.0221 PD, 1.0034 PD, 1.0064 PD, 1.0090 PD, 1.0112 PD, 1.0131 PD, 1.0147 PD,
 1.0160 PD, 1.0169 PD, 1.0176 PD, 1.0180 PD, 1.0025 PD, 1.0048 PD, 1.0068 PD,
 1.0086 PD, 1.0102 PD, 1.0115 PD, 1.0126 PD, 1.0135 PD, 1.0142 PD, 1.0147 PD,
 1.0150 PD, 1.0020 PD, 1.0037 PD, 1.0053 PD, 1.0068 PD, 1.0080 PD, 1.0091 PD,
 1.0101 PD, 1.0109 PD, 1.0115 PD, 1.0121 PD, 1.0124 PD, 1.0127 PD, 1.0015 PD,
 1.0029 PD, 1.0042 PD, 1.0054 PD, 1.0064 PD, 1.0074 PD, 1.0082 PD, 1.0089 PD,
 1.0095 PD, 1.0100 PD, 1.0104 PD, 1.0107 PD, 1.0109 PD, 1.0012 PD, 1.0024 PD,
 1.0034 PD, 1.0044 PD, 1.0052 PD, 1.0060 PD, 1.0067 PD, 1.0073 PD, 1.0079 PD}
```

Discarding kth nearest impact

In[69]:=
```
PDHat = Table[N[Integrate[((k - 1) * PD) / (rk ^ 3) * ((x)! / (((k - y2) - 1)! * (x - (k - y2))!)) *
      ((rk - CubeRoot[y2]) / (CubeRoot[y1] - CubeRoot[y2])) ^ ((k - y2) - 1) *
      (1 - ((rk - CubeRoot[y2]) / (CubeRoot[y1] - CubeRoot[y2]))) ^ (x - (k - y2)) *
      (1 / (CubeRoot[y1] - CubeRoot[y2])), {rk, CubeRoot[y2], CubeRoot[y1]}], 5], {k, 2, 100}]
```

Out[70]=
```
{0.69336 PD, 0.96150 PD, 0.83841 PD, 0.93904 PD, 0.98800 PD, 0.90080 PD, 0.94776 PD,
 0.97795 PD, 0.99449 PD, 0.93313 PD, 0.95843 PD, 0.97692 PD, 0.98950 PD, 0.99696 PD,
 0.95197 PD, 0.96703 PD, 0.97888 PD, 0.98782 PD, 0.99415 PD, 0.99813 PD, 0.96387 PD,
 0.97352 PD, 0.98147 PD, 0.98785 PD, 0.99279 PD, 0.99639 PD, 0.99876 PD, 0.97185 PD,
 0.97838 PD, 0.98394 PD, 0.98859 PD, 0.99238 PD, 0.99537 PD, 0.99761 PD, 0.99913 PD,
 0.97747 PD, 0.98208 PD, 0.98611 PD, 0.98957 PD, 0.99250 PD, 0.99491 PD, 0.99685 PD,
 0.99833 PD, 0.99937 PD, 0.98156 PD, 0.98494 PD, 0.98794 PD, 0.99057 PD, 0.99286 PD,
 0.99481 PD, 0.99644 PD, 0.99776 PD, 0.99878 PD, 0.99953 PD, 0.98463 PD, 0.98718 PD,
 0.98947 PD, 0.99152 PD, 0.99333 PD, 0.99490 PD, 0.99626 PD, 0.99741 PD,
 0.99835 PD, 0.99909 PD, 0.99964 PD, 0.98700 PD, 0.98896 PD, 0.99075 PD,
 0.99237 PD, 0.99382 PD, 0.99511 PD, 0.99624 PD, 0.99722 PD, 0.99805 PD,
 0.99874 PD, 0.99930 PD, 0.99971 PD, 0.98886 PD, 0.99041 PD, 0.99183 PD,
 0.99313 PD, 0.99430 PD, 0.99537 PD, 0.99631 PD, 0.99715 PD, 0.99788 PD,
 0.99850 PD, 0.99902 PD, 0.99945 PD, 0.99977 PD, 0.99035 PD, 0.99159 PD,
 0.99273 PD, 0.99379 PD, 0.99476 PD, 0.99564 PD, 0.99644 PD, 0.99715 PD, 0.99779 PD}
```

Discarding half kth nearest impact

In[71]:=
```
PDHat = Table[N[Integrate[((k - (1/2)) * PD) / (rk ^ 3) * ((x)! / (((k - y2) - 1)! * (x - (k - y2))!)) *
    ((rk - CubeRoot[y2]) / (CubeRoot[y1] - CubeRoot[y2])) ^ ((k - y2) - 1) *
    (1 - ((rk - CubeRoot[y2]) / (CubeRoot[y1] - CubeRoot[y2]))) ^ (x - (k - y2)) *
    (1 / (CubeRoot[y1] - CubeRoot[y2])), {rk, CubeRoot[y2], CubeRoot[y1]}], 5], {k, 2, 100}]
```

Out[71]=
```
{1.0400 PD, 1.2019 PD, 0.97814 PD, 1.0564 PD, 1.0868 PD, 0.97587 PD, 1.0155 PD, 1.0391 PD,
 1.0497 PD, 0.97979 PD, 1.0020 PD, 1.0176 PD, 1.0276 PD, 1.0326 PD, 0.98370 PD,
 0.99725 PD, 1.0077 PD, 1.0153 PD, 1.0203 PD, 1.0231 PD, 0.98682 PD, 0.99564 PD,
 1.0028 PD, 1.0084 PD, 1.0126 PD, 1.0155 PD, 1.0173 PD, 0.98921 PD, 0.99525 PD,
 1.0003 PD, 1.0045 PD, 1.0079 PD, 1.0105 PD, 1.0123 PD, 1.0134 PD, 0.99104 PD,
 0.99536 PD, 0.99908 PD, 1.0023 PD, 1.0049 PD, 1.0070 PD, 1.0087 PD, 1.0099 PD,
 1.0107 PD, 0.99246 PD, 0.99564 PD, 0.99845 PD, 1.0009 PD, 1.0030 PD, 1.0048 PD,
 1.0062 PD, 1.0074 PD, 1.0082 PD, 1.0088 PD, 0.99358 PD, 0.99599 PD, 0.99815 PD,
 1.0001 PD, 1.0017 PD, 1.0032 PD, 1.0044 PD, 1.0055 PD, 1.0063 PD, 1.0069 PD,
 1.0073 PD, 0.99447 PD, 0.99634 PD, 0.99804 PD, 0.99956 PD, 1.0009 PD, 1.0021 PD,
 1.0032 PD, 1.0041 PD, 1.0048 PD, 1.0054 PD, 1.0059 PD, 1.0062 PD, 0.99520 PD,
 0.99667 PD, 0.99803 PD, 0.99926 PD, 1.0004 PD, 1.0014 PD, 1.0022 PD, 1.0030 PD,
 1.0037 PD, 1.0042 PD, 1.0047 PD, 1.0051 PD, 1.0053 PD, 0.99579 PD, 0.99698 PD,
 0.99807 PD, 0.99908 PD, 1.0000 PD, 1.0008 PD, 1.0016 PD, 1.0022 PD, 1.0028 PD}
```

B. Proof for the estimate of irradiance (first approximation)

The estimate of irradiance is:

$$\hat{I}(r_k) = \frac{kI(P)}{r_k n}$$

We want to find out the radius r_k. The photons are stratified, so each photon is situated in one subdivision. After projecting all photons on the positive side of the interval, two photons are in each subdivision. A and B indicate the borders of one subdivision inside the scene. $[A, B]$ is length of one subdivision inside the scene.

For even n / even k, odd n / odd k:

$$\begin{aligned}
\hat{I}(r_k) &= \frac{kI(P)}{\frac{2}{3}[A,B]n} \\
&= \frac{kI(P)}{(A + (B-A)\frac{2}{3})n} \\
&= \frac{kI(P)}{((\frac{k}{2}-1)\frac{2}{n} + (\frac{k}{2}\frac{2}{n} - (\frac{k}{2}-1)\frac{2}{n})\frac{2}{3})n} \\
&= \frac{kI(P)}{\frac{3k-2}{3n}n}
\end{aligned}$$

For even n / odd k, odd n / even k:

$$\begin{aligned}
\hat{I}(r_k) &= \frac{kI(P)}{\frac{1}{3}[A,B]n} \\
&= \frac{kI(P)}{(A + (B-A)\frac{1}{3})n} \\
&= \frac{kI(P)}{((\frac{k+1}{2}-1)\frac{2}{n} + (\frac{k+1}{2}\frac{2}{n} - (\frac{k+1}{2}-1)\frac{2}{n})\frac{1}{3})n} \\
&= \frac{kI(P)}{\frac{3k-1}{3n}n}
\end{aligned}$$